Chapter 9: Feynman Rules for QED

In Chapter 8 we learned how the QED interaction,

$$\mathcal{L}_{\text{int}} = -e\,\bar{\psi}\gamma^{\mu}\psi A_{\mu},$$

enters the S-matrix through the Dyson expansion. In principle, every scattering amplitude can be obtained by expanding

$$S = T\exp\left[i\int d^4x\,\mathcal{L}_{\text{int}}(x)\right],$$

in powers of e, applying Wick's theorem, evaluating all contractions, and Fourier transforming the result. In practice, doing this from the beginning for every process would be unbearably slow.

Feynman rules are the compressed form of that calculation. They are not an additional physical postulate. They are a bookkeeping language for perturbation theory. A Feynman diagram records which fields are contracted, which external particles are present, how momentum flows, and which algebraic factors must be multiplied. This diagrammatic method was introduced in its modern form by Feynman and connected to the covariant S-matrix expansion by Dyson in the early development of QED (Feynman 1949; Dyson 1949).

The purpose of this chapter is to derive and use the momentum-space Feynman rules for QED. By the end, you should be able to look at a process such as

$$e^-\mu^- \to e^-\mu^-, \qquad e^-e^+ \to \gamma\gamma, \qquad e^-\gamma \to e^-\gamma,$$

draw the contributing diagrams at a given order in e, and translate them into amplitudes with the correct spinors, propagators, polarization vectors, momentum-conservation factors, and signs.

Throughout we use natural units,

hbar=c=1,

and the mostly-minus metric,

$$\eta_{\mu\nu}=\mathrm{diag}(1,-1,-1,-1).$$

Repeated Lorentz indices are summed.

9.1 What a Feynman rule is

A scattering calculation in perturbative QED has three layers.

First, there is the physical process. For example,

$$e^-(p_1)+e^+(p_2)\to\mu^-(p_3)+\mu^+(p_4),$$

means that an electron and a positron are prepared with four-momenta p_1,p_2, and a muon and antimuon are detected with four-momenta p_3,p_4.

Second, there is a term in the Dyson expansion. At second order in e, for example,

$$\frac{i^2}{2!}\int d^4x\,d^4y\,T[\mathcal{L}_{\rm int}(x)\mathcal{L}_{\rm int}(y)]\,.$$

Third, there is a diagrammatic representation of the Wick contractions. A line in a Feynman diagram represents either an external particle state or a contraction between two fields. A vertex represents one insertion of the interaction

$$-e\,\bar{\psi}\gamma^\mu\psi\,A_\mu\,.$$

A Feynman rule is the algebraic factor associated with one of these graphical elements.

For example, the QED vertex contains one photon field A_μ, one Dirac field ψ, and one conjugate Dirac field $\bar{\psi}$. Therefore the basic QED vertex connects one photon line and two fermion lines:

$$\text{fermion}\quad\longleftrightarrow\quad\text{fermion}+\text{photon}.$$

In momentum space, that vertex contributes the factor

$$-ie\gamma^\mu.$$

The factor -ie comes from imathcal L_{int}, and the matrix γ^μ comes from the Dirac current barψγ^μψ.

The diagram is therefore a compact way of remembering a term in the operator expansion. This is the essential meaning of Feynman rules.

9.2 The QED Lagrangian with gauge fixing

For perturbative calculations we separate the Lagrangian into a free part and an interaction part. In a covariant gauge, we take

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - e\bar{\psi}\gamma^\mu\psi A_\mu.$$

Here xi is a gauge parameter. It labels a family of covariant gauge choices:

- xi=1: Feynman gauge;
- xi=0: Landau gauge;
- general xi: covariant Rxi-type gauge.

The gauge-fixing term does not change physical observables. It changes the photon propagator, which affects intermediate algebra but not gauge-invariant S-matrix elements. The reason is that the unphysical longitudinal and time-like photon components cancel from physical amplitudes. This cancellation is organized by Ward identities, which will be the subject of Chapter 11. Standard QFT treatments of covariant gauge fixing and QED perturbation theory are given, for example, in Peskin and Schroeder and in Weinberg (Peskin and Schroeder 1995; Weinberg 1995).

The interaction is

$$\mathcal{L}_{\text{int}} = -e\bar{\psi}\gamma^\mu\psi A_\mu.$$

The free part determines the propagators. The interaction part determines the vertex.

9.3 Conventions for amplitudes and states

Before listing rules, we must fix normalization conventions. We use covariantly normalized one-particle states:

$$\langle \mathbf{p}', s' | \mathbf{p}, s \rangle = (2\pi)^3 2E_{\mathbf{p}}\, \delta^{(3)}(\mathbf{p} - \mathbf{p}')\delta_{ss'}.$$

For photons,

$$\langle \mathbf{k}', \lambda' | \mathbf{k}, \lambda \rangle = (2\pi)^3 2\omega_{\mathbf{k}}\, \delta^{(3)}(\mathbf{k} - \mathbf{k}')\delta_{\lambda\lambda'}.$$

With these conventions, the S-matrix element is written as

$$\langle f|S|i\rangle = \langle f|i\rangle + i(2\pi)^4\delta^{(4)}(P_f - P_i)\,\mathcal{M}_{fi}.$$

The quantity mathcal M(fi) is called the invariant amplitude or matrix element. The adjective “invariant” means that, once external spin and polarization labels are handled correctly, mathcal M is the Lorentz-covariant object from which cross sections and decay rates are built.

Many authors instead write the diagrammatic product directly as imathcal M. We will often do this:

$$\text{diagram} = i\mathcal{M}.$$

This convention is useful because each propagator and vertex naturally carries factors of i.

9.4 Momentum-space propagators

A propagator is the Fourier transform of a time-ordered two-point function. It represents the propagation of an internal virtual particle between two interaction vertices.

The word virtual does not mean “unphysical” in a careless sense. It means that the internal line is not an asymptotic particle detected in the far past or far future. Its four-momentum is integrated over and need not satisfy the on-shell relation $p^2=m^2$. Only external particles are required to be on shell.

Electron propagator

For the Dirac field,

$$S_F(x-y) = \langle 0|T\psi(x)\bar{\psi}(y)|0\rangle.$$

In momentum space,

S□F□(p) = (i(slashed p+m)) □ (p□-m□+iε),

where

slashed p equiv γ^μ p_μ .

Thus an internal electron or positron line carrying momentum p contributes

boxed (i(slashed p+m)) □ (p□-m□+iε)

to the amplitude.

The iε prescription tells us how the poles are displaced in the complex energy plane. It encodes time ordering and the causal boundary condition appropriate to the Feynman propagator.

Photon propagator

In a general covariant gauge,

$$D_{\mu\nu}(k) = \frac{-i}{k^2+i\epsilon}\left[\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2+i\epsilon}\right].$$

Thus an internal photon line carrying momentum k contributes

$$\boxed{\frac{-i}{k^2+i\epsilon}\left[\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2+i\epsilon}\right]}$$

with Lorentz indices μ,nu attached to the two vertices connected by the photon.

In Feynman gauge, xi=1, this becomes especially simple:

$$\boxed{D_{\mu\nu}(k) = \frac{-i\eta_{\mu\nu}}{k^2 + i\epsilon}}$$

which is why Feynman gauge is often the first choice for explicit calculations.

9.5 The QED vertex

Each interaction insertion contributes

$$i \int d^4x \, \mathcal{L}_{\text{int}}(x) = -ie \int d^4x \, \bar{\psi}(x)\gamma^\mu \psi(x) A_\mu(x).$$

After Fourier transforming the fields, the integral over x gives a momentum-conservation delta function at the vertex. Removing the overall delta function associated with total momentum conservation, the local vertex contributes

$$\boxed{-ie\gamma^\mu}$$

where μ is the Lorentz index of the photon line.

The vertex always connects:

1. one photon line;
2. one fermion line entering the vertex;
3. one fermion line leaving the vertex.

The fermion line carries an arrow. This arrow tracks the flow of fermion number. For electrons, the arrow points in the same direction as particle flow. For positrons, the arrow points opposite to particle flow.

This convention is not decorative. It tells us how to multiply spinor matrices along a fermion line.

9.6 External electron and positron factors

Internal lines are represented by propagators. External lines are represented by wavefunctions.

For a spin-frac12 particle, the external wavefunctions are the Dirac spinors u,bar u,v,bar v. They solve the free Dirac equations

(slashed p-m)u(p,s)=0,

(slashed p+m)v(p,s)=0.

The spinor u(p,s) describes an external electron, while v(p,s) describes an external positron. The bar denotes the Dirac adjoint,

$$\bar{u} = u^\dagger \gamma^0, \qquad \bar{v} = v^\dagger \gamma^0.$$

The external-line rules are:

$$\boxed{\text{incoming electron: } u(p,s)}$$

$$\boxed{\text{outgoing electron: } \bar{u}(p,s)}$$

$$\boxed{\text{incoming positron: } \bar{v}(p,s)}$$

$$\boxed{\text{outgoing positron: } v(p,s)}$$

A good way to remember this is to follow the fermion arrow. Along a fermion line, the spinor chain is read opposite to the direction in which bras and kets are written but consistently along the arrow flow. For example, in electron scattering,

$$e^-(p) \to e^-(p'),$$

a photon coupling produces the spinor factor

$$\bar{u}(p')(-ie\gamma^\mu)u(p).$$

The outgoing electron gives bar u(p'), the incoming electron gives u(p), and the vertex lies between them.

For positron scattering,

$$e^+(p) \to e^+(p'),$$

the corresponding current is

$$\bar{v}(p)(-ie\gamma^\mu)v(p').$$

The order looks reversed because the fermion arrow for a positron points opposite to the direction of positron momentum.

9.7 External photon factors

A physical external photon has momentum k^μ and polarization label λ. Its polarization vector is denoted

$$\varepsilon^\mu(k, \lambda).$$

For an incoming photon, the external factor is

$$\boxed{\varepsilon^\mu(k, \lambda)}$$

or, depending on index placement at the vertex,

$$\varepsilon_\mu(k, \lambda).$$

For an outgoing photon, the external factor is complex conjugated:

$$\boxed{\varepsilon^{\mu *}(k, \lambda)}$$

or

$$\varepsilon_\mu^*(k, \lambda).$$

Physical photon polarizations are transverse:

$$k_\mu \varepsilon^\mu(k, \lambda) = 0.$$

Because the photon is massless and gauge redundant, the polarization vector is not unique. Replacing

$$\varepsilon^\mu(k, \lambda) \to \varepsilon^\mu(k, \lambda) + c\, k^\mu$$

for any scalar c should not change a physical amplitude. This is another practical form of gauge invariance. If a calculated amplitude changes under this replacement, something has gone wrong unless the object being calculated is not itself a physical observable.

9.8 Momentum conservation and loop integration

At each vertex, four-momentum is conserved. If momenta p_1,p_2,p_3 enter a vertex, the vertex contributes

$$(2\pi)^4 \delta^{(4)}(p_1 + p_2 + p_3)$$

before the overall momentum-conservation delta function is removed.

In practical amplitude calculations, we usually do the following:

1. assign momenta to all external lines;
2. assign momenta to internal lines;
3. impose momentum conservation at each vertex;
4. remove the overall factor

$$(2\pi)^4 \delta^{(4)}(P_f - P_i);$$

5. write the remaining product as imathcal M.

When a diagram contains an undetermined internal momentum, we integrate over it:

$$\int \frac{d^4\ell}{(2\pi)^4}.$$

Such an integral is called a loop integral. A loop is a closed path of internal propagators in the diagram. Loop diagrams will become central in Chapters 14–16, but the rule already belongs here:

$$\boxed{\text{for each independent loop momentum } \ell, \quad \int \frac{d^4\ell}{(2\pi)^4}.}$$

For example, a one-loop correction to the photon propagator contains an internal electron-positron loop. Its momentum is not fixed by external kinematics, so it is integrated over.

9.9 The full momentum-space Feynman rules for QED

We now collect the rules in one place.

Use these rules to compute imathcal M, with the overall factor

$$(2\pi)^4\delta^{(4)}(P_f - P_i)$$

removed.

Internal lines

An internal fermion line with momentum p:

boxed (i(slashed p+m)) □ (p□-m□+iε)

An internal photon line with momentum k, in covariant gauge:

$$\boxed{\frac{-i}{k^2+i\epsilon}\left[\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2+i\epsilon}\right]}$$

In Feynman gauge:

$$\boxed{\frac{-i\eta_{\mu\nu}}{k^2 + i\epsilon}}$$

Vertices

Each QED vertex:

$$\boxed{-ie\gamma^\mu}$$

where μ is contracted with the photon line attached to the vertex.

External fermions

Incoming electron:

$$\boxed{u(p, s)}$$

Outgoing electron:

$$\boxed{\bar{u}(p, s)}$$

Incoming positron:

$$\boxed{\bar{v}(p, s)}$$

Outgoing positron:

$$\boxed{v(p, s)}$$

External photons

Incoming photon:

$$\boxed{\varepsilon^{\mu}(k,\lambda)}$$

Outgoing photon:

$$\boxed{\varepsilon^{\mu *}(k,\lambda)}$$

Momentum integrals

For each independent loop momentum:

$$\boxed{\int \frac{d^4\ell}{(2\pi)^4}}$$

Delta functions

Impose momentum conservation at every vertex. Remove the final overall factor

$$(2\pi)^4\delta^{(4)}(P_f - P_i)$$

when defining mathcal M.

Fermion signs

Include a factor

$$\boxed{-1}$$

for each closed fermion loop.

Also include minus signs required by exchanging identical external fermions. These signs come from the anticommutation of fermionic creation and annihilation operators.

Symmetry factors

Divide by the diagram’s symmetry factor when a diagram has nontrivial automorphisms: that is, when interchanging identical internal lines or vertices leaves the diagram unchanged. For most tree-level QED scattering amplitudes with labeled external particles, the symmetry factor is 1. Vacuum diagrams and some loop diagrams require more care.

The origin of symmetry factors is the same as in Wick's theorem. The expansion of the exponential gives a factor 1/n!, while the number of equivalent contractions often cancels it. Any leftover overcounting becomes the symmetry factor.

9.10 Reading fermion lines

Fermion lines require more attention than scalar lines because each segment carries a matrix or spinor. The final answer must be a Lorentz scalar, but intermediate objects are spinor chains.

Consider an electron line passing through several photon vertices. Suppose an incoming electron with momentum p scatters into an outgoing electron with momentum p' through two internal propagators and three photon vertices. A typical spinor chain might look like

bar u(p') (-ieγ□μ□□) (i(slashed q□+m)) □ (q□□-m□+iε) (-ieγ□μ□□) (i(slashed q□+m)) □ (q□□-m□+iε) (-ieγ□μ□□) u(p).

The matrices are ordered along the fermion line. Matrix order matters because gamma matrices do not generally commute:

$$\gamma^\mu \gamma^\nu \neq \gamma^\nu \gamma^\mu.$$

This is one of the most common sources of mistakes in QED calculations. A reliable habit is:

1. choose a fermion arrow;
2. begin with the spinor at one end;
3. multiply every vertex and propagator in the order encountered;
4. end with the spinor at the other end.

For a continuous open fermion line, the result is a spinor bilinear such as

$$\bar{u}(\cdots)u, \qquad \bar{v}(\cdots)v, \qquad \bar{v}(\cdots)u, \qquad \bar{u}(\cdots)v.$$

For a closed fermion loop, there are no external spinors. The gamma matrices form a trace:

tr ≤ft[γ□μ□□ (i(slashed q□+m)) □ (q□□-m□+iε) γ□μ□□ (i(slashed q□+m)) □ (q□□-m□+iε) ⋯].

Then multiply by the closed-fermion-loop factor -1.

9.11 Example: electron-muon scattering

Although the basic QED Lagrangian written earlier used one Dirac field ψ, physical QED may contain several charged Dirac fields. For electrons and muons,

$$\mathcal{L}_{\text{int}} = -e\,\bar{\psi}_e\gamma^\mu\psi_e A_\mu - e\,\bar{\psi}_\mu\gamma^\mu\psi_\mu A_\mu,$$

with different masses m_e and m_μ. The photon couples to both because both carry electric charge -e.

Consider

$$e^-(p_1) + \mu^-(p_2) \to e^-(p_3) + \mu^-(p_4).$$

At lowest order, one photon is exchanged between the electron and muon currents. Define the momentum transfer

$$q = p_3 - p_1 = p_2 - p_4.$$

In Feynman gauge, the diagram gives

$$i\mathcal{M} = [\bar{u}_e(p_3)(-ie\gamma^\mu)u_e(p_1)]\,\frac{-i\eta_{\mu\nu}}{q^2 + i\epsilon}\,[\bar{u}_\mu(p_4)(-ie\gamma^\nu)u_\mu(p_2)]\,.$$

Multiplying the factors,

$$i\mathcal{M} = i\,\frac{e^2}{q^2 + i\epsilon}\,[\bar{u}_e(p_3)\gamma^\mu u_e(p_1)]\,[\bar{u}_\mu(p_4)\gamma_\mu u_\mu(p_2)]\,.$$

Thus

$$\boxed{\mathcal{M} = \frac{e^2}{q^2 + i\epsilon}\,[\bar{u}_e(p_3)\gamma^\mu u_e(p_1)]\,[\bar{u}_\mu(p_4)\gamma_\mu u_\mu(p_2)]}$$

with the understanding that for physical scattering q^2 is usually spacelike, so $q^2<0$.

This example shows the basic architecture of QED amplitudes:

$$\text{charged current} \times \text{photon propagator} \times \text{charged current.}$$

In the nonrelativistic limit, the same photon exchange is responsible for the Coulomb force. In the relativistic theory, the Coulomb interaction is not inserted separately; it arises from photon exchange.

9.12 Example: electron-positron annihilation into a muon pair

Now consider

$$e^-(p_1) + e^+(p_2) \to \mu^-(p_3) + \mu^+(p_4).$$

At lowest order, the electron and positron annihilate into a virtual photon, which then produces the muon pair. Let

$$q = p_1 + p_2 = p_3 + p_4.$$

In Feynman gauge,

$$i\mathcal{M} = [\bar{v}_e(p_2)(-ie\gamma^\mu)u_e(p_1)]\,\frac{-i\eta_{\mu\nu}}{q^2 + i\epsilon}\,[\bar{u}_\mu(p_3)(-ie\gamma^\nu)v_\mu(p_4)]\,.$$

Therefore

$$\boxed{\mathcal{M} = \frac{e^2}{q^2 + i\epsilon}\left[\bar{v}_e(p_2)\gamma^\mu u_e(p_1)\right]\left[\bar{u}_\mu(p_3)\gamma_\mu v_\mu(p_4)\right]}$$

up to the standard convention that the full S-matrix element contains

$$i(2\pi)^4\delta^{(4)}(p_1 + p_2 - p_3 - p_4)\mathcal{M}.$$

This process is an example of an s-channel diagram. The name comes from the Mandelstam invariant

$$s = (p_1 + p_2)^2.$$

The virtual photon carries the total incoming momentum, so its denominator is s+iε.

By contrast, electron-muon scattering above was a t-channel process, because the virtual photon carried momentum transfer

$$q = p_3 - p_1,$$

and the corresponding Mandelstam invariant is

$$t = q^2.$$

The labels s,t,u are not new physics. They are a standard way to classify the kinematic channel through which momentum flows.

9.13 Example: Compton scattering

Compton scattering is

$$e^-(p) + \gamma(k) \to e^-(p') + \gamma(k').$$

At order e^2, there are two diagrams. The reason is simple: the electron line must absorb one photon and emit one photon, but the two photon vertices can occur in two possible orders.

Let the incoming photon have polarization εμ(k), and the outgoing photon have polarization εnu^(k').

The first diagram has intermediate electron momentum

$$p + k.$$

The second has intermediate electron momentum

$$p - k'.$$

The amplitude is

imathcal M = (-ie)□ bar u(p') ≤ft[γ^nu (i(slashed p+slashed k+m)) □ ((p+k)□-m□+iε) γ^μ + γ^μ (i(slashed p-slashed k'+m)) □ ((p-k')□-m□+iε) γ^nu] u(p) ε_μ(k)ε_nu^*(k').

This expression illustrates several important rules at once.

First, the two diagrams must be added coherently at the amplitude level. We do not square one diagram and then square the other separately. Quantum amplitudes interfere.

Second, the gamma matrices appear in the order encountered along the electron line. Interchanging them would change the amplitude.

Third, gauge invariance gives a powerful check. If we replace

$$\varepsilon_\mu(k) \to k_\mu$$

or

$$\varepsilon^*_\nu(k') \to k'_\nu,$$

the full sum of the two diagrams vanishes after using the on-shell Dirac equations. Usually neither diagram vanishes by itself. Gauge invariance requires the sum.

This is a pattern repeated throughout QED: physical consistency often becomes visible only after all diagrams of the same perturbative order are included.

9.14 Example: pair annihilation into two photons

Consider

$$e^-(p_1) + e^+(p_2) \to \gamma(k_1) + \gamma(k_2).$$

At order e², there are again two diagrams. The electron and positron annihilate, and the two photons can be attached in two possible orders along the fermion line.

Let the outgoing photon polarizations be

$$\varepsilon_\mu^*(k_1), \qquad \varepsilon_\nu^*(k_2).$$

One convenient expression is

imathcal M = (-ie)□ bar v(p□) ≤ft[γ^nu (i(slashed p□-slashed k□+m)) □ ((p□-k□)□-m□+iε) γ^μ + γ^μ (i(slashed p□-slashed k□+m)) □ ((p□-k□)□-m□+iε) γ^nu] u(p□) ε_μ^*(k□) ε_nu^*(k□).

The two terms are required because photons are identical bosons. Interchanging the two final photons gives another valid contraction. Since photons are bosons, the two contributions add with a plus sign.

If the final particles had been identical fermions, exchanging them would introduce a relative minus sign. This is exactly what happens in Møller scattering,

$$e^-e^- \to e^-e^-,$$

where the final electrons are indistinguishable and the amplitude contains two diagrams with a relative minus sign from Fermi statistics.

9.15 Identical particles and relative signs

Identical particles require careful treatment because labels in a diagram are not always physically meaningful.

For example, in Møller scattering,

$$e^-(p_1) + e^-(p_2) \to e^-(p_3) + e^-(p_4),$$

there are two tree-level photon-exchange diagrams:

1. $p_1 \to p_3$ and $p_2 \to p_4$;
2. $p_1 \to p_4$ and $p_2 \to p_3$.

The amplitude has the schematic form

$$\mathcal{M} = \mathcal{M}_t - \mathcal{M}_u.$$

The minus sign is not optional. It follows from the anticommutation of fermion operators. If one forgets it, the prediction violates the exchange antisymmetry required for identical fermions.

For identical bosons, by contrast, exchanged diagrams add with a plus sign. In pair annihilation,

$$e^-e^+ \to \gamma\gamma,$$

the two photon-ordering diagrams add.

There is a related but distinct issue in cross sections: if the final state contains identical particles, the phase-space integral must not count the same physical final state more than once. Therefore, for two identical final photons one includes a factor 1/2! in the phase-space calculation. This factor belongs to the probability or cross section, not to the amplitude itself as a relative sign.

9.16 Gauge choices in practical calculations

The covariant photon propagator contains the gauge parameter xi:

$$D_{\mu\nu}(k) = \frac{-i}{k^2 + i\epsilon}\left[\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2 + i\epsilon}\right].$$

The second term is longitudinal, proportional to kμ knu. If the photon couples to conserved currents, this longitudinal part often vanishes after contraction.

For example, in electron-muon scattering the photon propagator is contracted with

$$J_e^\mu = \bar{u}_e(p_3)\gamma^\mu u_e(p_1),$$

and

$$J_\mu^\nu = \bar{u}_\mu(p_4)\gamma^\nu u_\mu(p_2).$$

The momentum transfer is

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