Chapter 8: Perturbation Theory and the S-Matrix

In Chapter 7 we built the QED Lagrangian,

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi - e\,\bar{\psi}\gamma^{\mu}\psi A_{\mu},$$

up to the gauge-fixing terms needed for practical photon quantization. The new term,

$$\mathcal{L}_{\text{int}} = -e\,\bar{\psi}\gamma^{\mu}\psi A_{\mu},$$

is the interaction between the charged Dirac field and the electromagnetic field. It says that the electromagnetic potential Aμ couples to the conserved Dirac current

$$j^{\mu} = \bar{\psi}\gamma^{\mu}\psi.$$

This chapter explains how that interaction becomes a calculational machine.

The central difficulty is simple to state. We know how to quantize free fields exactly. We know the free electron, positron, and photon states. But the full interacting QED Hamiltonian cannot be diagonalized exactly in any useful closed form. The method that makes QED predictive is perturbation theory: we treat the interaction as small and compute physical quantities as a power series in the electric charge e, or equivalently in the fine-structure constant

$$\alpha = \frac{e^2}{4\pi}.$$

Since α simeq 1/137 at low energies, this expansion is remarkably effective for many electromagnetic processes.

The object that organizes scattering predictions is the S-matrix, or scattering matrix. It maps states prepared in the distant past to states observed in the distant future. In symbols,

$$|\text{out}\rangle = S|\text{in}\rangle.$$

This deceptively compact equation contains the bridge from quantum fields to measurable scattering cross sections.

Our goals in this chapter are to build that bridge carefully. We will introduce the interaction picture, derive the Dyson expansion, define time ordering, prove the working content of Wick's theorem, understand vacuum normalization and connected diagrams, and finally see how correlation functions become scattering amplitudes through the LSZ reduction idea.

Throughout we use natural units,

hbar=c=1,

and the mostly-minus metric,

η_{μnu}=diag(1,-1,-1,-1).

Repeated Lorentz indices are summed.

8.1 The problem of interacting time evolution

In ordinary quantum mechanics, a state evolves according to the Schrödinger equation,

$$i\frac{d}{dt}|\Psi(t)\rangle = H|\Psi(t)\rangle.$$

If the Hamiltonian is split into a solvable part and an interaction,

$$H = H_0 + H_{\text{int}},$$

then H_0 describes the free theory, while H_{int} contains the forces.

In QED, H_0 describes free electrons, positrons, and photons. The interaction term allows processes such as

$$e^- \to e^- + \gamma,$$

as an intermediate virtual process, or

$$e^-\mu^- \to e^-\mu^-,$$

through photon exchange.

The exact time-evolution operator $U(t,t_0)$ satisfies

$$i\frac{\partial}{\partial t}U(t,t_0) = H\,U(t,t_0), \qquad U(t_0,t_0) = 1.$$

If H were time independent and exactly solvable, then

$$U(t,t_0) = e^{-iH(t-t_0)}.$$

But for interacting quantum fields, this expression is not practically useful. We instead reorganize the calculation so that the known free evolution is treated exactly and the interaction is expanded systematically.

This is the purpose of the interaction picture.

8.2 Schrödinger, Heisenberg, and interaction pictures

Quantum theory allows different but equivalent ways of distributing time dependence between states and operators.

In the Schrödinger picture, states carry time dependence and operators are usually time independent:

$$|\Psi_S(t)\rangle = e^{-iH(t-t_0)}|\Psi_S(t_0)\rangle.$$

In the Heisenberg picture, states are time independent and operators carry all time dependence:

$$\mathcal{O}_H(t) = e^{iH(t-t_0)}\mathcal{O}_S e^{-iH(t-t_0)}.$$

In the interaction picture, we split the Hamiltonian,

$$H = H_0 + H_{\text{int}},$$

and let operators evolve with H_0, while states evolve with the interaction.

For an operator mathcal O,

$$\mathcal{O}_I(t) = e^{iH_0(t-t_0)}\mathcal{O}_S e^{-iH_0(t-t_0)}.$$

Thus the fields appearing in perturbation theory are free fields:

$$\psi_I(x), \qquad A_{I\mu}(x).$$

They satisfy the free Dirac and Maxwell equations, and they have the mode expansions developed in earlier chapters. The interaction-picture state satisfies

$$i\frac{d}{dt}|\Psi_I(t)\rangle = H_I(t)|\Psi_I(t)\rangle,$$

where

$$H_I(t) = e^{iH_0(t-t_0)}H_{\text{int}}e^{-iH_0(t-t_0)}.$$

For QED, assuming the usual interaction has no derivative coupling that would complicate the relation between mathcal L_{int} and H_{int},

$$H_I(t) = \int d^3x\, e\,\bar{\psi}_I(x)\gamma^\mu\psi_I(x)A_{I\mu}(x).$$

Equivalently,

$$H_I(t) = -\int d^3x\, \mathcal{L}_{\text{int}}(x),$$

with

$$\mathcal{L}_{\text{int}}(x) = -e\,\bar{\psi}_I(x)\gamma^\mu\psi_I(x)A_{I\mu}(x).$$

From now on, we usually suppress the subscript I, with the understanding that perturbative fields are free interaction-picture fields.

8.3 The time-evolution operator in the interaction picture

Define the interaction-picture evolution operator $U_{(I)}(t,t_0)$ by

$$|\Psi_I(t)\rangle = U_I(t,t_0)|\Psi_I(t_0)\rangle.$$

It satisfies

$$i\frac{\partial}{\partial t}U_I(t,t_0) = H_I(t)U_I(t,t_0), \qquad U_I(t_0,t_0) = 1.$$

Equivalently,

$$\frac{\partial}{\partial t}U_I(t,t_0) = -iH_I(t)U_I(t,t_0).$$

Integrating once gives

$$U_I(t,t_0) = 1 - i\int_{t_0}^{t} dt_1\, H_I(t_1)U_I(t_1,t_0).$$

Now insert the same equation recursively on the right-hand side. This produces the perturbative expansion

$$U_I(t,t_0) = 1 - i\int_{t_0}^{t} dt_1\, H_I(t_1) + (-i)^2\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H_I(t_1)H_I(t_2) + \cdots .$$

The second-order term has the operators ordered with later times to the left. This motivates a compact notation.

8.4 Time ordering

The time-ordering operator T rearranges operators so that later times stand to the left of earlier times. For two bosonic operators A(x) and B(y),

$$T\{A(x)B(y)\} = \begin{cases} A(x)B(y), & x^0 > y^0, \\ B(y)A(x), & y^0 > x^0. \end{cases}$$

For fermionic operators, time ordering includes a minus sign whenever two fermionic operators are exchanged. For example,

$$T\{\psi_\alpha(x)\bar{\psi}_\beta(y)\} = \begin{cases} \psi_\alpha(x)\bar{\psi}_\beta(y), & x^0 > y^0, \\ -\bar{\psi}_\beta(y)\psi_\alpha(x), & y^0 > x^0. \end{cases}$$

The minus sign is not optional. It comes from the anticommutation of fermionic fields and is ultimately responsible for the characteristic signs of closed fermion loops in Feynman diagrams.

Using time ordering, the interaction-picture evolution operator becomes the Dyson series,

$$U_I(t, t_0) = T \exp\left[-i \int_{t_0}^{t} dt'\, H_I(t')\right].$$

For QED this is often written covariantly as

$$U_I(t, t_0) = T \exp\left[i \int_{t_0}^{t} dt' \int d^3x\, \mathcal{L}_{\text{int}}(x)\right].$$

Taking $t_0 \to -\infty$ and $t \to +\infty$, the S-matrix is formally

$$S = T \exp\left[i \int d^4x\, \mathcal{L}_{\text{int}}(x)\right].$$

For QED,

$$S = T \exp\left[-ie \int d^4x\, \bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x)\right].$$

This time-ordered exponential formulation was central to Dyson’s covariant organization of QED perturbation theory (Dyson 1949).

8.5 First look at the expansion in QED

Expanding the QED S-matrix gives

$$S = 1 + (-ie)\int d^4x\, T\{\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x)\}$$

$$+\frac{(-ie)^2}{2!}\int d^4x\, d^4y\, T\{\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x)\bar{\psi}(y)\gamma^\nu\psi(y)A_\nu(y)\} + \cdots .$$

Each factor

$$-ie\gamma^\mu$$

will become a QED vertex in momentum-space Feynman rules.

The first-order term contains one photon field and two fermion fields. It can contribute, for example, to matrix elements involving one incoming electron, one outgoing electron, and one photon:

$$e^- \to e^- + \gamma.$$

However, for real on-shell particles in empty space, the process

$$e^- \to e^- + \gamma$$

is forbidden by energy-momentum conservation. The same vertex nevertheless appears inside higher-order processes, where the intermediate photon or electron may be off shell.

The second-order term contains two interaction vertices. It is the first order that can describe electron-muon scattering,

$$e^-(p)\mu^-(k) \to e^-(p')\mu^-(k'),$$

through the exchange of one virtual photon. Schematically,

$$\mathcal{M} \sim (-ie)^2 \left[\bar{u}(p')\gamma^\mu u(p)\right] \frac{-i\eta_{\mu\nu}}{q^2 + i\epsilon} \left[\bar{u}(k')\gamma^\nu u(k)\right],$$

where

$$q = p' - p = -(k' - k)$$

is the momentum transferred by the photon. The precise sign and gauge-dependent numerator of the photon propagator will be derived in Chapter 9. The important point here is structural: powers of e count interaction vertices.

8.6 Propagators as time-ordered two-point functions

A two-point function is a vacuum expectation value of two fields. The most important two-point functions in perturbation theory are time ordered. They are called Feynman propagators.

For a scalar field, the Feynman propagator is

$$\Delta_F(x-y) = \langle 0|T\{\phi(x)\phi(y)\}|0\rangle.$$

It measures the amplitude for a disturbance created at one spacetime point to be removed at another, with the correct time-ordering prescription for relativistic quantum theory. Feynman's spacetime formulation made this propagator language central to QED calculations (Feynman 1949).

For the Dirac field,

$$S_F(x-y)_{\alpha\beta} = \langle 0|T\{\psi_\alpha(x)\bar{\psi}_\beta(y)\}|0\rangle.$$

In momentum space,

S□F□(x-y) = ∫(d□p) □ ((2π)□) (i(slashed p+m)) □ (p□-m□+iε) e□-ip□(x-y)□.

Here

slashed p = γ^μ p_μ .

The iε prescription tells us how to pass the poles in the complex p^0-plane. It encodes causal propagation in the time-ordered sense.

For the photon field in Feynman gauge,

$$D_F^{\mu\nu}(x-y) = \langle 0|T\{A^\mu(x)A^\nu(y)\}|0\rangle = \int \frac{d^4k}{(2\pi)^4}\frac{-i\eta^{\mu\nu}}{k^2+i\epsilon}e^{-ik\cdot(x-y)}.$$

Other gauges modify the numerator but not the gauge-invariant physical predictions.

These propagators are the basic contractions used in Wick's theorem.

8.7 Normal ordering and contractions

To compute terms in the Dyson expansion, we must evaluate vacuum expectation values of products of free fields. Directly multiplying creation and annihilation operators is possible but inefficient. Wick's theorem provides the efficient method.

First we need two definitions.

A product of operators is normal ordered if all creation operators are placed to the left of all annihilation operators. For a scalar field,

$$:\phi(x)\phi(y):$$

means that, after writing each field in terms of creation and annihilation operators, every creation operator has been moved to the left.

The vacuum expectation value of any normal-ordered product of free fields vanishes unless no operators remain:

$$\langle 0| :\mathcal{O}: |0\rangle = 0$$

for any nontrivial normal-ordered product :mathcal O:.

A contraction is the difference between a time-ordered product and its normal-ordered product. For a scalar field,

$$\text{contraction}\varphi(x)\varphi\ \varphi(x)\varphi(y) \text{ equiv } T\backslash\varphi(x)\varphi(y)\backslash - :\varphi(x)\varphi(y): .$$

For free fields this contraction is exactly the Feynman propagator:

$$\text{contraction}\varphi(x)\varphi\ \varphi(x)\varphi(y) = \langle 0|T\backslash\varphi(x)\varphi(y)\backslash|0\rangle = \Delta_F(x-y).$$

For Dirac fields,

$$\text{contraction}\psi\_\alpha(x)\text{bar}\psi\_\beta\ \psi\_\alpha(x)\text{bar}\psi\_\beta(y) = S_F(x-y)_{\alpha\beta}.$$

For photons in Feynman gauge,

$$\text{contraction}A\^\mu(x)A\^\text{nu}\ A\^\mu(x)A\^\text{nu}(y) = D_F^{\mu\text{nu}}(x-y).$$

Contractions are not physical particle trajectories. They are algebraic replacements inside time-ordered products. Their diagrammatic representation as lines is the origin of Feynman diagrams.

8.8 Wick's theorem

Wick's theorem states that a time-ordered product of free fields equals the normal-ordered product plus all possible contractions, with the correct signs for fermions. Wick introduced this method in his 1950 analysis of collision-matrix calculations (Wick 1950).

For a scalar field, the theorem begins as

$$T\backslash\varphi_1\varphi_2\backslash = :\varphi_1\varphi_2: + \text{contraction}\varphi_1\varphi_2\ \varphi_1\varphi_2\ ,$$

where $\varphi_i=\varphi(x_i)$.

For four scalar fields,

$$T\{\phi_1\phi_2\phi_3\phi_4\} =: \phi_1\phi_2\phi_3\phi_4 :$$

+ contractionφ□φ□ φ□φ□:φ□φ□: + contractionφ□φ□ φ□φ□:φ□φ□: + contractionφ□φ□ φ□φ□:φ□φ□:

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Taking the vacuum expectation value removes every term that still contains a normal-ordered nontrivial operator. Thus

$$\langle 0|T\{\phi_1\phi_2\phi_3\phi_4\}|0\rangle = \Delta_F(x_1 - x_2)\Delta_F(x_3 - x_4)$$

$$+\Delta_F(x_1 - x_3)\Delta_F(x_2 - x_4) + \Delta_F(x_1 - x_4)\Delta_F(x_2 - x_3).$$

This is the simplest example of a major principle:

> In a free Gaussian quantum field theory, all higher vacuum correlation functions are built from two-point functions.

For fermions the same idea holds, but every exchange of fermionic operators contributes a minus sign. For example,

$$\langle 0|T\{\psi_\alpha(x)\bar{\psi}_\beta(y)\psi_\gamma(z)\bar{\psi}_\delta(w)\}|0\rangle$$

equals a sum of products of fermion propagators, with signs determined by the permutation needed to bring contracted fermion fields together.

In practical QED calculations, Wick's theorem converts the Dyson expansion into Feynman diagrams.

8.9 A simple scalar example before QED

Before applying the idea to QED, it is useful to look at a scalar model with interaction

$$\mathcal{L}_{\text{int}} = -\frac{\lambda}{4!}\phi^4.$$

The S-matrix is

$$S = T\exp\left[-i\int d^4x\,\frac{\lambda}{4!}\phi^4(x)\right].$$

Consider the four-point correlation function,

$$G^{(4)}(x_1, x_2, x_3, x_4) = \langle 0|T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)S\}|0\rangle.$$

At first order in λ,

$$G^{(4)}_{\lambda}(x_1, x_2, x_3, x_4) = -\frac{i\lambda}{4!}\int d^4z\,\langle 0|T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\phi^4(z)\}|0\rangle.$$

One class of Wick contractions connects each external field $\varphi(x_i)$ to one of the four fields at z. There are 4! such contractions, canceling the 4! in the interaction. This gives

$$G^{(4)}_{\lambda,\text{connected}} = -i\lambda\int d^4z\,\Delta_F(x_1 - z)\Delta_F(x_2 - z)\Delta_F(x_3 - z)\Delta_F(x_4 - z).$$

Diagrammatically, this is one four-valent vertex at z, with four external lines attached.

This example teaches three lessons that will persist in QED:

1. interaction terms become vertices;
2. propagators come from contractions;
3. combinatorial factors arise from counting equivalent contractions.

In QED, the vertex has one photon line and two fermion lines because the interaction contains

$$\bar{\psi}\gamma^\mu\psi A_\mu.$$

8.10 QED contractions and the shape of the vertex

The QED interaction density is

$$\mathcal{L}_{\text{int}}(x) = -e\,\bar{\psi}(x)\gamma^{\mu}\psi(x)A_{\mu}(x).$$

Each interaction point contains three field factors:

$$\bar{\psi}, \qquad \psi, \qquad A_{\mu}.$$

Therefore each QED vertex connects:

1. one fermion line entering the vertex;
2. one fermion line leaving the vertex;
3. one photon line.

The gamma matrix γ^μ sits at the vertex and connects the Lorentz index of the photon to the spinor indices of the fermion line.

For example, the second-order term in the S-matrix is

$$S^{(2)} = \frac{(-ie)^2}{2}\int d^4x\,d^4y\,T\{\bar{\psi}(x)\gamma^{\mu}\psi(x)A_{\mu}(x)\bar{\psi}(y)\gamma^{\nu}\psi(y)A_{\nu}(y)\}.$$

A contraction of the two photon fields gives

contractionA_μ(x)A_nu A_μ(x)A_nu(y) = D□Fμnu□(x-y).

The remaining fermion fields can be contracted with external electron and muon states, or with source fields inside a correlation function. This is the algebraic origin of one-photon exchange.

A useful mental picture is:

$$\text{two QED vertices} + \text{one photon contraction} \quad \Longrightarrow \quad \text{electromagnetic scattering.}$$

This is not yet the full Feynman-rule machinery, but it is already the skeleton of Chapter 9.

8.11 Correlation functions

A correlation function is a vacuum expectation value of a time-ordered product of fields. For example,

$$G^{(2)}_{\alpha\beta}(x,y) = \langle\Omega|T\{\psi_\alpha(x)\bar{\psi}_\beta(y)\}|\Omega\rangle$$

is the exact interacting two-point function of the electron field. Here $|\Omega\rangle$ denotes the true interacting vacuum, not the free vacuum $|0\rangle$.

Similarly, a four-point function might be

$$G^{(4)}_{\alpha\beta\gamma\delta}(x_1,x_2,x_3,x_4) = \langle\Omega|T\{\psi_\alpha(x_1)\bar{\psi}_\beta(x_2)\psi_\gamma(x_3)\bar{\psi}_\delta(x_4)\}|\Omega\rangle.$$

Correlation functions are more general than scattering amplitudes. They contain information about propagation, interactions, bound-state poles, and response to sources. Scattering amplitudes are extracted from special singular parts of correlation functions when external particles are placed on shell.

In perturbation theory, we express interacting-vacuum correlation functions in terms of free-vacuum expectation values. This requires one more important ingredient: vacuum normalization.

8.12 The interacting vacuum and the Gell-Mann–Low formula

The free vacuum $|0\rangle$ is the state with no free particles. The interacting vacuum $|\Omega\rangle$ is the lowest-energy state of the full interacting theory.

They are not exactly the same. For example, the interacting QED vacuum includes virtual fluctuations such as electron-positron pairs and photons. This statement should not be interpreted as a literal gas of particles in the vacuum; rather, it means that the interacting ground state is not annihilated by the same free creation and annihilation operators that define $|0\rangle$.

The Gell-Mann–Low formula relates interacting correlation functions to free-vacuum expectation values with an inserted time-evolution operator (Gell-Mann and Low 1951). In the form used in perturbation theory,

$$\langle\Omega|T\{\mathcal{O}_1\cdots\mathcal{O}_n\}|\Omega\rangle = \frac{\langle 0|T\{\mathcal{O}_1\cdots\mathcal{O}_n \exp[i\int d^4x\,\mathcal{L}_{\rm int}(x)]\}|0\rangle}{\langle 0|T\{\exp[i\int d^4x\,\mathcal{L}_{\rm int}(x)]\}|0\rangle}.$$

The denominator is crucial. It removes vacuum bubbles: diagrams that are completely disconnected from the operator insertions.

A vacuum bubble is a closed diagram with no external attachment. For example, in a scalar theory, contractions inside the interaction terms alone can form a diagram that contributes to

$$\langle 0|S|0\rangle.$$

Such a factor changes the vacuum-to-vacuum amplitude but has no effect on the scattering process being studied. The denominator divides it out.

In QED, vacuum bubbles include closed electron loops connected by photon lines. They contribute to the vacuum persistence amplitude but cancel from normalized correlation functions unless they are connected to the external fields.

8.13 Connected and disconnected diagrams

A Feynman diagram is connected if every part of it is linked to every other part by propagator lines and vertices. It is disconnected if it splits into independent pieces.

For example, in a four-point scalar correlation function, one contribution is

$$\Delta_F(x_1-x_2)\Delta_F(x_3-x_4).$$

This is disconnected: it describes propagation from x_1 to x_2 and separately from x_3 to x_4, with no interaction between the two pairs.

A connected contribution might be

$$-i\lambda\int d^4z\,\Delta_F(x_1-z)\Delta_F(x_2-z)\Delta_F(x_3-z)\Delta_F(x_4-z),$$

where all four external points meet at the same interaction vertex.

Connected diagrams matter because scattering is about interaction. If two incoming particles simply pass through without influencing each other, that contribution belongs to the identity part of the S-matrix or to disconnected products of lower processes. The nontrivial scattering amplitude is extracted from the connected part.

This is why one usually writes

$$S = 1 + iT,$$

where the identity 1 represents no scattering and T contains the transition operator. For momentum eigenstates,

$$\langle f|iT|i\rangle = i(2\pi)^4 \delta^{(4)}(P_f - P_i)\mathcal{M}_{i\to f}.$$

The quantity mathcal M(i→ f) is the invariant matrix element or scattering amplitude. It is the central output of Feynman-diagram calculations.

The delta function expresses total energy-momentum conservation. The amplitude mathcal M contains the dynamical information.

8.14 From amplitudes to probabilities

The S-matrix element

$$\langle f|S|i\rangle$$

is a quantum amplitude. Probabilities are obtained by taking absolute squares, with the correct normalization of states and integration over final phase space.

For a 2→ n process,

$$a(p_1) + b(p_2) \to 1(k_1) + \cdots + n(k_n),$$

the differential cross section has the general structure

$$d\sigma = \frac{1}{\text{flux}} |\mathcal{M}|^2 d\Pi_n,$$

where $d\Pi_n$ is the Lorentz-invariant n-body phase-space measure,

$$d\Pi_n = (2\pi)^4 \delta^{(4)} \left(p_1 + p_2 - \sum_{j=1}^{n} k_j \right) \prod_{j=1}^{n} \frac{d^3 k_j}{(2\pi)^3 2E_j}.$$

For two incoming particles with four-momenta p_1, p_2, masses m_1, m_2, and relativistic normalization, the invariant flux factor is often written as

$$4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}.$$

Thus,

$$d\sigma = \frac{|\mathcal{M}|^2}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} d\Pi_n.$$

This formula will become a working tool in Chapter 10. For now, the important conceptual chain is:

$$\text{Lagrangian} \longrightarrow S \longrightarrow \mathcal{M} \longrightarrow |\mathcal{M}|^2 \longrightarrow \text{cross sections and decay rates.}$$

8.15 The LSZ bridge: from correlation functions to scattering

We have now introduced two kinds of objects:

1. correlation functions, such as

$$\langle \Omega | T\{\phi(x_1) \cdots \phi(x_n)\} | \Omega \rangle;$$

2. S-matrix elements, such as

$$\langle f|S|i\rangle.$$

The LSZ reduction formula, named after Lehmann, Symanzik, and Zimmermann, explains how to obtain S-matrix elements from time-ordered correlation functions by isolating the external one-particle poles (Lehmann, Symanzik, and Zimmermann 1955).

The basic idea is easiest for a scalar field. Suppose φ(x) creates a one-particle state from the vacuum with nonzero overlap:

$$\langle\Omega|\phi(0)|p\rangle \neq 0.$$

The exact two-point function then has a pole near

$$p^2 = m^2,$$

corresponding to the propagation of a physical one-particle state. In momentum space, the exact propagator behaves as

$$\tilde{G}^{(2)}(p) \sim \frac{iZ}{p^2 - m^2 + i\epsilon}$$

near the pole. The constant Z is called the field-strength renormalization. It measures the overlap between the field φ and the physical one-particle state.

The LSZ prescription says, roughly, that to extract a scattering amplitude, one should:

1. Fourier transform the relevant time-ordered correlation function;
2. multiply by inverse external propagator factors, such as p_i^2-m^2;
3. take the external momenta on shell;
4. include appropriate powers of $Z^{(-1/2)}$.

For a scalar 2→2 process, schematically,

$$\langle p_3 p_4, \mathrm{out}|p_1 p_2, \mathrm{in}\rangle_{\mathrm{connected}}$$

is obtained from

$$\tilde{G}^{(4)}(p_1, p_2, p_3, p_4)$$

by amputating the four external propagators and taking

$$p_i^2 \to m^2.$$

To amputate an external line means to remove the external propagator associated with it. Diagrammatically, one cuts off the propagator legs attached to the outside of the diagram, leaving the interacting core.

In QED the same principle applies, with modifications appropriate to spinors and photons. External electron lines are associated with spinors u(p) or bar u(p), external positron lines with v(p) or bar v(p), and external photons with polarization vectors ε^μ(p). The detailed rules will be derived in Chapter 9.

There is also a subtlety: because the photon is massless, QED has infrared issues in the definition of perfectly exclusive charged-particle scattering states. The perturbative S-matrix remains an essential tool, but physically measurable predictions require inclusive treatment of sufficiently soft radiation. We will study this in Chapter 17.

8.16 External particles and amputated diagrams

In practical Feynman-diagram calculations, the LSZ logic is packaged into external-line rules.

For example, an incoming electron contributes a spinor

$$u(p, s),$$

where p is its momentum and s labels spin. An outgoing electron contributes

$$\bar{u}(p, s).$$

An incoming positron contributes

$$\bar{v}(p, s),$$

and an outgoing positron contributes

$$v(p, s).$$

An external photon contributes a polarization vector,

$$\varepsilon^{\mu}(p, \lambda)$$

or its complex conjugate depending on whether the photon is incoming or outgoing.

The internal lines of a diagram are propagators. The external lines are not propagators in the final amplitude; they have been amputated and replaced by wavefunctions or polarization vectors.

For instance, the electron-muon scattering amplitude has the schematic form

$$\mathcal{M} = \left[\text{electron current}\right] \times \left[\text{photon propagator}\right] \times \left[\text{muon current}\right].$$

More explicitly, in Feynman gauge,

$$i\mathcal{M} = \bar{u}(p')(-ie\gamma^{\mu})u(p)\frac{-i\eta_{\mu\nu}}{q^2 + i\epsilon}\bar{u}(k')(-ie\gamma^{\nu})u(k).$$

Therefore,

$$\mathcal{M} = -\frac{e^2}{q^2}\left[\bar{u}(p')\gamma^{\mu}u(p)\right]\left[\bar{u}(k')\gamma_{\mu}u(k)\right],$$

up to the conventional treatment of the iε away from the pole and with the sign depending

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