Chapter 7: Building the QED Lagrangian

The previous chapters gave us the two free quantum fields that QED must join together:

1. the Dirac field $\psi(x)$, whose quanta are spin-frac12 charged particles and antiparticles;
2. the electromagnetic four-potential $A\mu(x)$, whose physical transverse excitations are photons.

A free electron field cannot emit a photon. A free photon field cannot scatter from an electron. To describe light and matter as they are observed, we need an interaction. The central claim of this chapter is that the interaction of QED is not chosen arbitrarily. It is strongly constrained, and essentially determined at the minimal renormalizable level, by the demand that a local phase convention for the charged field be physically meaningless.

That demand is called local U(1) gauge invariance. It is one of the simplest and most powerful ideas in theoretical physics. Historically, the connection between electromagnetism and local phase transformations was clarified by Weyl in 1929, after the development of quantum mechanics and the Dirac equation (Weyl 1929). In modern language, QED is the quantum field theory obtained by coupling the Dirac field to a U(1) gauge field.

Throughout this chapter we use natural units,

$$\text{hbar}=c=1,$$

and the mostly-minus metric,

$$\eta\mu\text{nu}=\mathrm{diag}(1,-1,-1,-1).$$

Repeated Lorentz indices are summed.

7.1 The free starting point

Before introducing interactions, recall the free Dirac Lagrangian,

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi,$$

where

$$\bar{\psi} = \psi^\dagger \gamma^0.$$

The Euler-Lagrange equation is the Dirac equation,

$$(i\gamma^\mu \partial_\mu - m)\psi = 0,$$

which Dirac introduced as a relativistic first-order wave equation for the electron (Dirac 1928). In quantum field theory, ψ is promoted to an operator-valued spinor field, and its mode expansion contains both particle and antiparticle creation and annihilation operators.

The free electromagnetic field is described by

$$\mathcal{L}_{\text{EM}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

The tensor $F_{(\mu nu)}$ contains the electric and magnetic fields. The potential Aμ has gauge redundancy:

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \lambda(x),$$

where λ(x) is an arbitrary smooth function. The field strength is unchanged because partial derivatives commute:

$$F_{\mu\nu} \to \partial_\mu (A_\nu + \partial_\nu \lambda) - \partial_\nu (A_\mu + \partial_\mu \lambda) = F_{\mu\nu}.$$

Thus the free theory

$$\mathcal{L}_0 = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

describes noninteracting charged Dirac particles and noninteracting photons. QED begins when we ask how these two fields can be coupled without destroying Lorentz invariance, locality, unitarity, and gauge redundancy.

7.2 Global phase symmetry of the Dirac field

The free Dirac Lagrangian has a simple symmetry:

$$\psi(x) \to e^{-iq\alpha}\psi(x), \qquad \bar{\psi}(x) \to \bar{\psi}(x)e^{iq\alpha},$$

where α is a constant real number and q is a charge parameter.

This is called a global U(1) symmetry.

Let us unpack the terminology.

The group U(1) is the group of complex numbers of unit magnitude,

$$e^{i\theta}, \qquad \theta \in \mathbb{R}.$$

Multiplication is the group operation. The word global means that the same phase rotation is applied at every spacetime point. The phase parameter α does not depend on x.

The free Dirac Lagrangian is invariant because the phases cancel:

$$\bar{\psi}\psi \to \bar{\psi}e^{iq\alpha}e^{-iq\alpha}\psi = \bar{\psi}\psi,$$

and, since α is constant,

$$\partial_\mu(e^{-iq\alpha}\psi) = e^{-iq\alpha}\partial_\mu\psi.$$

Therefore

$$\bar{\psi}i\gamma^\mu\partial_\mu\psi \to \bar{\psi}e^{iq\alpha}i\gamma^\mu e^{-iq\alpha}\partial_\mu\psi = \bar{\psi}i\gamma^\mu\partial_\mu\psi.$$

By Noether's theorem, every continuous global symmetry of the action gives a conserved current under suitable assumptions on the fields and boundary terms (Noether 1918). For the global phase symmetry of the Dirac field, the conserved current is

$$j^\mu(x) = q\,\bar{\psi}(x)\gamma^\mu\psi(x).$$

Its conservation law is

$$\partial_\mu j^\mu = 0.$$

The associated conserved charge is

$$Q = \int d^3x\, j^0(x) = q\int d^3x\, \psi^\dagger(x)\psi(x),$$

with normal ordering understood in the quantum theory.

This is the first appearance of electric charge in this chapter. In the quantum Dirac field, particles and antiparticles carry opposite values of this conserved charge. With the usual mode expansion, the charge operator takes the schematic form

$$Q = q\sum_s \int \frac{d^3p}{(2\pi)^3}\left[b_s^\dagger(\mathbf{p})b_s(\mathbf{p}) - d_s^\dagger(\mathbf{p})d_s(\mathbf{p})\right],$$

up to the chosen normalization of one-particle states. Thus the b^† excitations carry charge q, while the d^† excitations carry charge -q. This particle-antiparticle charge structure is one of the basic consequences of quantizing the Dirac field (Peskin and Schroeder 1995).

For an electron field, one often writes

$$q = -e, \qquad e > 0,$$

so that the electron has charge -e and the positron has charge +e. Many QED formulas depend only on e^2, but signs matter in amplitudes, so it is important to keep track of conventions.

7.3 Why global symmetry is not enough

A global phase symmetry says that the absolute phase of ψ is not physically meaningful. But it still compares phases at different spacetime points using a fixed convention. Local gauge invariance asks for something stronger:

$$\psi(x) \to e^{-iq\lambda(x)}\psi(x),$$

where λ(x) is now an arbitrary function of spacetime.

This is called a local U(1) transformation.

The word local means that the phase rotation may be chosen independently at different spacetime points. For example, one may rotate the phase near the origin by one amount and the phase near a distant detector by another amount.

At first this seems harmless. The mass term remains invariant:

$$\bar{\psi}\psi \to \bar{\psi}e^{iq\lambda(x)}e^{-iq\lambda(x)}\psi = \bar{\psi}\psi.$$

But the kinetic term is not invariant. The derivative acts not only on ψ, but also on the spacetime-dependent phase:

$$\partial_\mu\psi \to \partial_\mu\left(e^{-iq\lambda(x)}\psi\right) = e^{-iq\lambda(x)}\left(\partial_\mu\psi - iq(\partial_\mu\lambda)\psi\right).$$

Therefore

$$\bar{\psi}i\gamma^\mu\partial_\mu\psi$$

acquires an extra term. Explicitly,

$$\bar{\psi}i\gamma^\mu\partial_\mu\psi \to \bar{\psi}i\gamma^\mu\partial_\mu\psi + q(\partial_\mu\lambda)\bar{\psi}\gamma^\mu\psi.$$

The free Dirac Lagrangian is not locally U(1)-invariant.

This failure is extremely instructive. It tells us that ordinary derivatives compare field values at neighboring points using a rigid phase convention. If the phase convention can vary from point to point, the ordinary derivative is no longer a meaningful object by itself.

A simple analogy is useful. Suppose you want to compare compass directions at two nearby locations on a curved surface. You need a rule for transporting a direction from one point to the other. Without such a rule, the statement "these two arrows point in the same direction" is incomplete. In gauge theory, the gauge field supplies the rule for comparing internal phases at nearby spacetime points.

7.4 The covariant derivative

To restore local invariance, we replace the ordinary derivative by a covariant derivative.

A covariant derivative is a derivative modified so that it transforms in the same way as the field it differentiates. For a charged Dirac field, define

$$D_\mu = \partial_\mu + iqA_\mu.$$

We want

$$D_\mu\psi \to e^{-iq\lambda(x)} D_\mu\psi.$$

This will hold if the gauge field transforms as

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu\lambda(x).$$

Indeed,

$$D'_\mu\psi' = (\partial_\mu + iqA'_\mu)\left(e^{-iq\lambda}\psi\right).$$

Substitute $A_\mu' = A_\mu + \partial_\mu\lambda$:

$$D'_\mu\psi' = (\partial_\mu + iqA_\mu + iq\partial_\mu\lambda)\left(e^{-iq\lambda}\psi\right).$$

Now differentiate:

$$D'_\mu \psi' = e^{-iq\lambda} \left[\partial_\mu \psi - iq(\partial_\mu \lambda)\psi + iqA_\mu \psi + iq(\partial_\mu \lambda)\psi \right].$$

The two terms involving $\partial_\mu \lambda$ cancel, leaving

$$D'_\mu \psi' = e^{-iq\lambda}(\partial_\mu + iqA_\mu)\psi = e^{-iq\lambda} D_\mu \psi.$$

This is the key mathematical mechanism of QED.

The ordinary derivative failed because it differentiated the local phase. The gauge field A_μ transforms in precisely the way needed to cancel that extra derivative of the phase.

The locally invariant Dirac Lagrangian is therefore

$$\mathcal{L}_{\text{matter}} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi.$$

Expanding the covariant derivative gives

$$\mathcal{L}_{\text{matter}} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - q\,\bar{\psi}\gamma^\mu \psi\, A_\mu.$$

The second term,

$$\mathcal{L}_{\text{int}} = -q\,\bar{\psi}\gamma^\mu \psi\, A_\mu,$$

is the QED interaction.

This procedure is called minimal coupling. In this context, minimal coupling means that we introduce the electromagnetic interaction by replacing

$$\partial_\mu \longrightarrow D_\mu = \partial_\mu + iqA_\mu$$

and add no additional higher-dimension interactions.

The word “minimal” matters. Gauge invariance alone does not forbid every other possible term. For example, an operator of the form

$$\bar{\psi}\sigma^{\mu\nu}\psi F_{\mu\nu}, \qquad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu],$$

is gauge invariant and Lorentz invariant, but in four spacetime dimensions it has mass dimension five. It is therefore not part of the minimal renormalizable QED Lagrangian. Such terms naturally appear in effective field theory, where they are suppressed by a high mass scale. We will return to this viewpoint in Chapter 23. The distinction between renormalizable minimal interactions and higher-dimension effective interactions is standard in modern quantum field theory (Weinberg 1995).

7.5 The field strength from the covariant derivative

The electromagnetic field strength can be recovered directly from the covariant derivative.

Compute the commutator:

$$[D_\mu,D_\nu]\psi = D_\mu D_\nu\psi-D_\nu D_\mu\psi.$$

Using

$$D_\mu=\partial_\mu+iqA_\mu,$$

and remembering that A_μ acts by multiplication, one finds

$$[D_\mu,D_\nu]\psi = iq(\partial_\mu A_\nu-\partial_\nu A_\mu)\psi.$$

Therefore

$$[D_\mu,D_\nu]\psi = iqF_{\mu\nu}\psi,$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

This equation gives a useful interpretation of $F_{\mu\nu}$. The field strength measures the failure of two covariant derivatives to commute. In geometric language, it is a curvature associated with the gauge connection A_μ. For QED, the gauge group is Abelian, so the expression for $F_{\mu\nu}$ is especially simple. In non-Abelian gauge theories, such as Yang-Mills theory, the field strength contains additional terms quadratic in the gauge fields.

Since $F_{\mu\nu}$ is invariant under

$$A_\mu \to A_\mu + \partial_\mu \lambda,$$

the photon kinetic term

$$-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

is locally gauge invariant.

This term is also Lorentz invariant and local. It contains two derivatives of A_μ in the equations of motion, as expected for a massless spin-1 field. The gauge-invariant kinetic term for electromagnetism is the same structure that appears in classical Maxwell theory, now understood as part of a quantum field theory.

7.6 The QED Lagrangian

Combining the locally invariant matter term and the electromagnetic kinetic term gives the QED Lagrangian:

$$\boxed{\mathcal{L}_{\text{QED}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi}$$

with

$$D_\mu = \partial_\mu + iqA_\mu.$$

Expanded out,

$$\boxed{\mathcal{L}_{\text{QED}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - q\,\bar{\psi}\gamma^\mu \psi A_\mu}$$

This is the central formula of the chapter.

For an electron field, one may set q=-e, with e>0. Then

$$D_\mu = \partial_\mu - ieA_\mu,$$

and the interaction term becomes

$$\mathcal{L}_{\text{int}} = +e\,\bar{\psi}\gamma^\mu\psi A_\mu.$$

Other texts often make slightly different sign choices for q, Aμ, or the definition of the gauge transformation. Physical predictions are unchanged when all signs are treated consistently. In later scattering calculations, the important rule is to keep one convention fixed from the Lagrangian to the Feynman rules.

Let us identify the pieces:

-frac14F□μnu□F□μnu□

is the free photon term;

$$\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

is the free electron-positron Dirac term;

$$-q\,\bar{\psi}\gamma^\mu\psi A_\mu$$

is the interaction between the electromagnetic field and the charged Dirac current.

This Lagrangian is:

- Lorentz invariant, because all Lorentz indices are contracted;
- local, because fields are multiplied at the same spacetime point;
- gauge invariant, because it is built from Dμψ, barψ, and F(μnu);
- renormalizable in four spacetime dimensions, because all operators have mass dimension at most four.

The mass dimensions are worth checking. In four dimensions, the Lagrangian density has mass dimension four. From the kinetic terms,

$$[\psi] = \frac{3}{2}, \qquad [A_\mu] = 1, \qquad [F_{\mu\nu}] = 2.$$

The interaction operator has dimension

$$[\bar{\psi}\gamma^\mu \psi A_\mu] = \frac{3}{2} + \frac{3}{2} + 1 = 4.$$

Therefore q is dimensionless:

$$[q] = 0.$$

This dimensionless coupling is one reason QED is perturbatively renormalizable. The systematic treatment of divergences and renormalization will begin in Chapters 14 and 15.

7.7 Why a photon mass term is absent

One might ask why the Lagrangian does not contain

$$\frac{1}{2} m_\gamma^2 A_\mu A^\mu.$$

This would be a Lorentz-invariant mass term for the vector field. But it is not gauge invariant. Under

A_μ→ A_μ+∂_μλ,

we have

$$A_\mu A^\mu \to (A_\mu + \partial_\mu \lambda)(A^\mu + \partial^\mu \lambda).$$

This produces extra terms:

$$A_\mu A^\mu + 2A^\mu \partial_\mu \lambda + (\partial_\mu \lambda)(\partial^\mu \lambda).$$

These do not vanish for an arbitrary local function λ(x). Thus a photon mass term is forbidden by local U(1) gauge invariance.

This is not merely a formal statement. A massless vector field has two physical transverse polarization states. A massive vector field has three physical polarization states. Gauge redundancy is the mechanism that removes the unphysical components of Aμ and leaves the correct photon degrees of freedom.

In practical calculations, we will later add gauge-fixing terms such as

$$-\frac{1}{2\xi}(\partial_\mu A^\mu)^2.$$

These terms are not physical photon mass terms. They are calculational devices used to define propagators while preserving gauge-invariant predictions for observables.

7.8 Equations of motion

The QED equations of motion follow by varying the action with respect to barψ, ψ, and Aμ.

Variation with respect to barψ gives the Dirac equation in an electromagnetic field:

$$(i\gamma^\mu D_\mu - m)\psi = 0.$$

Expanded,

$$(i\gamma^\mu \partial_\mu - m)\psi - q\gamma^\mu A_\mu \psi = 0.$$

This equation says that the charged Dirac field propagates in the presence of the gauge potential Aμ.

Variation with respect to Aμ gives Maxwell's equation with a source:

$$\partial_\nu F^{\nu\mu} = j^\mu,$$

where

$$j^\mu = q\,\bar{\psi}\gamma^\mu\psi.$$

Thus the Dirac field is the source of the electromagnetic field.

Together,

$$(i\gamma^\mu D_\mu - m)\psi = 0,$$

$$\partial_\nu F^{\nu\mu} = q\,\bar{\psi}\gamma^\mu\psi,$$

are the classical field equations of QED. In the quantum theory, these become operator equations, subject to the subtleties of gauge fixing, renormalization, and composite operator definitions.

The conservation of the current follows immediately from Maxwell's equation:

$$\partial_\mu j^\mu = \partial_\mu\partial_\nu F^{\nu\mu}.$$

Because $F^{(\text{nu}\mu)}$ is antisymmetric,

$$F^{\nu\mu} = -F^{\mu\nu},$$

while ∂μ∂nu is symmetric under interchange of μ and nu. Therefore

$$\partial_\mu\partial_\nu F^{\nu\mu} = 0,$$

and hence

$$\partial_\mu j^\mu = 0.$$

The same conservation law also follows from the Dirac equation and its adjoint. This agreement is important: gauge invariance, the equations of motion, and charge conservation are not separate accidents. They are different expressions of the same structure.

7.9 Gauge symmetry versus physical symmetry

It is important to distinguish two ideas that

Document information

Chapter 7: Building the QED Lagrangian

Project	Quantum Electrodynamics
Document	Document 1.11
Author	terry.mart
Verifier	Not verified
Downloaded	July 04, 2026 22:24 KST
Status	Working
Document link	https://www.theorytrace.com/projects/quantum-electrodynamics/documents/chapter-7--building-the-qed-lagrangian/