Chapter 2: Relativistic Quantum Mechanics and Its Limits

The first route toward QED does not begin with fields. It begins with a natural question:

> Can we make an ordinary quantum-mechanical wave equation that is compatible with special relativity?

This chapter follows that question to its breaking point. We will meet the Klein-Gordon equation, the Dirac equation, conserved relativistic currents, negative-energy solutions, spin, antiparticles, and finally the reason why a fixed-particle-number theory cannot be the final language of relativistic quantum physics.

The lesson is subtle. The Klein-Gordon and Dirac equations are not wrong. They are indispensable. The Klein-Gordon equation correctly describes free relativistic spin-0 fields after reinterpretation, and the Dirac equation correctly describes spin-1/2 matter such as electrons. But when they are treated as one-particle wave equations in the same sense as the nonrelativistic Schrödinger equation, they reveal deep limitations. The cure is not to discard them; it is to promote the wavefunctions into quantum fields.

That promotion is the conceptual bridge from relativistic quantum mechanics to QED.

2.1 The relativistic energy-momentum relation

In nonrelativistic quantum mechanics, the free-particle Schrödinger equation is built from the classical energy formula

$$E = \frac{\mathbf{p}^2}{2m}.$$

The usual quantization rule replaces energy and momentum by differential operators,

$$E \longrightarrow i\hbar\frac{\partial}{\partial t}, \qquad \mathbf{p} \longrightarrow -i\hbar\nabla.$$

This gives

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi.$$

The wavefunction ψ(t,x) is a complex probability amplitude. Its squared magnitude $|\psi|^2$ is interpreted as a probability density for finding the particle near x at time t.

Special relativity changes the starting energy formula. For a free particle of rest mass m, energy E, and three-momentum p, the relativistic relation is

$$E^2 = \mathbf{p}^2c^2 + m^2c^4.$$

Equivalently, in natural units hbar=c=1, which we will use from now on unless stated otherwise,

$$E^2 = \mathbf{p}^2 + m^2.$$

This relation is often called the mass-shell condition. The word “mass-shell” means that the four-momentum

$$p^\mu = (E, \mathbf{p})$$

lies on the relativistic surface

$$p^\mu p_\mu = m^2,$$

where we use the metric convention

$$\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1).$$

Thus

$$p^\mu p_\mu = E^2 - \mathbf{p}^2.$$

A free relativistic particle satisfies

$$p^\mu p_\mu = m^2.$$

The challenge is to turn this into a quantum wave equation without breaking Lorentz invariance. Lorentz invariance means that the equation must have the same form in every inertial frame.

2.2 The Klein-Gordon equation

The most direct relativistic wave equation is obtained by applying the substitutions

$$E \to i\frac{\partial}{\partial t}, \qquad \mathbf{p} \to -i\nabla$$

to

$$E^2 - \mathbf{p}^2 = m^2.$$

This gives

$$-\frac{\partial^2 \phi}{\partial t^2} + \nabla^2 \phi = m^2 \phi.$$

It is more elegant to write this using the spacetime derivative

$$\partial_\mu = \frac{\partial}{\partial x^\mu}, \qquad \partial^\mu = \eta^{\mu\nu}\partial_\nu.$$

The d'Alembertian operator is

$$\Box = \partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \nabla^2.$$

Then the equation becomes

$$(\Box + m^2)\phi = 0.$$

This is the Klein-Gordon equation. It was developed in the early history of relativistic wave mechanics and became the standard relativistic equation for spin-0 particles, though its physical interpretation required later field-theoretic clarification (Pauli and Weisskopf 1934; Bjorken and Drell 1964).

The field φ(x) may be real or complex. If φ is real, it describes a neutral spin-0 degree of freedom in the later field interpretation. If φ is complex, it can describe charged spin-0 degrees of freedom.

Plane-wave solutions

A plane wave has the form

$$\phi(x) = e^{-ip\cdot x} = e^{-iEt+i\mathbf{p}\cdot\mathbf{x}},$$

where

$$p\cdot x = p_\mu x^\mu = Et - \mathbf{p}\cdot\mathbf{x}.$$

Substituting into the Klein-Gordon equation gives

$$(-p^2 + m^2)e^{-ip\cdot x} = 0,$$

so the wave is a solution when

$$p^2 = m^2.$$

In components,

$$E^2 - \mathbf{p}^2 = m^2.$$

Therefore

$$E = \pm\sqrt{\mathbf{p}^2 + m^2}.$$

Here we meet the first important surprise: the equation has both positive-energy and negative-energy solutions.

At the level of the equation alone, both signs are mathematically allowed. A relativistic second-order equation does not distinguish between them. This is already a warning that the naive one-particle interpretation will be difficult.

2.3 Conserved currents and the probability problem

A conserved current is a spacetime vector j^μ satisfying a continuity equation,

$$\partial_\mu j^\mu = 0.$$

Writing

$$j^\mu = (\rho, \mathbf{j}),$$

this becomes

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0.$$

The physical meaning is local conservation. If ρ is a density and j is the corresponding flow, then the amount of the conserved quantity inside a region changes only because current flows through the boundary.

For the Schrödinger equation, there is a conserved probability density

$$\rho = |\psi|^2,$$

which is always nonnegative. That nonnegativity is essential. A probability density cannot be negative.

The Klein-Gordon equation has a conserved current, but it is not suitable as a probability current in the same way. For a complex Klein-Gordon field, define

$$j^\mu = i\left(\phi^* \partial^\mu \phi - \phi\, \partial^\mu \phi^*\right).$$

One can verify directly that

$$\partial_\mu j^\mu = 0$$

whenever φ satisfies the Klein-Gordon equation.

Let us check the time component for a plane wave

$$\phi(x) = Ae^{-iEt+i\mathbf{p}\cdot\mathbf{x}}.$$

Then

$$\partial^0\phi = \frac{\partial\phi}{\partial t} = -iE\phi, \qquad \partial^0\phi^* = +iE\phi^*.$$

Thus

$$j^0 = i\left(\phi^*(-iE\phi) - \phi(iE\phi^*)\right) = 2E|A|^2.$$

For E>0, this is positive. For E<0, this is negative.

That is the problem. If j^0 were a probability density, a negative-energy solution would have negative probability density. That cannot be right.

The modern resolution is that the Klein-Gordon current is not a one-particle probability current. For a complex scalar field, it is a charge current. Its density may be positive or negative because charge may be positive or negative. A negative value does not mean negative probability; it means opposite charge. This reinterpretation is a field-theoretic one and was central to the later understanding of relativistic scalar particles (Pauli and Weisskopf 1934).

Example: why charge density may be negative

Suppose a theory contains two kinds of spin-0 particles: one with charge +q, one with charge -q. A density of positive particles gives positive charge density. A density of negative particles gives negative charge density. There is no contradiction because charge is not probability.

This example foreshadows the antiparticle idea. In a relativistic quantum theory, the “negative-energy sector” is not merely a nuisance. It is trying to tell us that the theory contains particles and antiparticles.

2.4 Why first order in time matters

The nonrelativistic Schrödinger equation is first order in time:

$$i\frac{\partial \psi}{\partial t} = H\psi.$$

If $\psi(t_0,x)$ is known at one time t_0, the equation determines its future evolution. This fits the standard quantum-mechanical structure in which the state at one time determines the state at later times.

The Klein-Gordon equation is second order in time:

$$\frac{\partial^2 \phi}{\partial t^2} = (\nabla^2 - m^2)\phi.$$

To determine a solution, one must specify both

$$\phi(t_0, \mathbf{x}) \quad \text{and} \quad \partial_t \phi(t_0, \mathbf{x}).$$

This is not automatically wrong. Classical wave equations are often second order in time. But for a one-particle quantum wavefunction, it complicates the usual probability interpretation.

This motivated Dirac to seek a relativistic equation that is first order in time and also Lorentz invariant. The result was the Dirac equation, introduced in 1928 (Dirac 1928).

2.5 The Dirac equation

Dirac wanted an equation linear in both time and space derivatives, with the schematic form

$$i\frac{\partial \psi}{\partial t} = \left(-i\boldsymbol{\alpha} \cdot \nabla + \beta m\right)\psi.$$

Here ψ is not a single complex number at each spacetime point. It is a multi-component object. The matrices α^i and β act on its components.

The equation must reproduce the relativistic energy relation

$$E^2 = \mathbf{p}^2 + m^2.$$

For this to happen, the Hamiltonian

$$H = \boldsymbol{\alpha} \cdot \mathbf{p} + \beta m$$

must satisfy

$$H^2 = \mathbf{p}^2 + m^2.$$

Expanding,

$$H^2 = (\alpha^i p_i + \beta m)(\alpha^j p_j + \beta m).$$

To make this equal to p^2+m^2 for arbitrary momentum, the matrices must obey

$$\{\alpha^i, \alpha^j\} = 2\delta^{ij},$$

$$\{\alpha^i, \beta\} = 0,$$

$$\beta^2 = 1.$$

The notation

$$\{A, B\} = AB + BA$$

is called the anticommutator.

These algebraic conditions cannot be satisfied by ordinary numbers. They require matrices. The smallest faithful representation in 3+1 spacetime dimensions uses 4×4 matrices. Therefore the wavefunction ψ has four complex components.

A four-component object transforming in the appropriate way under Lorentz transformations is called a Dirac spinor. The word “spinor” means that the object transforms under rotations and boosts differently from an ordinary vector. A vector returns to itself under a 2π rotation. A spin-1/2 spinor changes sign under a 2π rotation and returns to itself only after a 4π rotation. This is not a contradiction because physical probabilities are quadratic in amplitudes.

Covariant form

The Dirac equation is usually written in Lorentz-covariant form as

$$(i\gamma^\mu \partial_\mu - m)\psi = 0.$$

The matrices γ^μ satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}.$$

The adjoint spinor is

$$\bar{\psi} = \psi^\dagger \gamma^0.$$

With this notation, the Dirac equation is compact and manifestly suited to relativistic calculations. Much of Chapter 5 will be devoted to the practical technology of these matrices and spinors.

2.6 The Dirac conserved current

The Dirac equation has a conserved current

$$j^\mu = \bar{\psi}\gamma^\mu \psi.$$

Its time component is

$$j^0 = \bar{\psi}\gamma^0\psi = \psi^\dagger\gamma^0\gamma^0\psi = \psi^\dagger\psi.$$

Since

$$\psi^\dagger \psi = \sum_a |\psi_a|^2 \geq 0,$$

the Dirac density is nonnegative. This looks much better than the Klein-Gordon case if one is searching for a one-particle probability interpretation.

Indeed, at the level of free positive-energy wave packets, the Dirac equation behaves much like a relativistic one-particle equation. It also naturally describes spin-1/2 degrees of freedom and gives the electron a magnetic moment with g=2 at the lowest relativistic level, before radiative corrections are included (Dirac 1928; Bjorken and Drell 1964).

But the deeper problems have not disappeared. The Dirac equation also has negative-energy solutions.

2.7 Negative-energy solutions of the Dirac equation

For a plane wave

$$\psi(x) = u(p) e^{-ip\cdot x},$$

the Dirac equation becomes

$$(\gamma^\mu p_\mu - m) u(p) = 0.$$

Nontrivial solutions exist when

$$p^2 = m^2,$$

so again

$$E = \pm\sqrt{\mathbf{p}^2 + m^2}.$$

The Dirac equation did not remove negative-energy solutions. It made the probability density positive, but the spectrum still contains energies unbounded below if interpreted naively.

This creates a severe problem. In ordinary quantum mechanics, a system is stable because it has a lowest-energy state. If there were physical one-electron states with arbitrarily negative energy, an electron could radiate energy and fall forever into lower and lower states. Matter would be unstable.

Dirac's historical response was the hole theory. In simplified terms, he proposed that all negative-energy electron states are filled in the vacuum. Because of the Pauli exclusion principle, an ordinary electron cannot fall into an already occupied negative-energy state. A missing electron in this filled sea behaves like a positively charged particle. This idea was part of the conceptual route to antiparticles, and Dirac later discussed the possibility of a positively charged counterpart to the electron (Dirac 1931). The positron was experimentally discovered by Anderson in cosmic-ray tracks in 1932, with the discovery paper published in 1933 (Anderson 1933).

Modern QFT does not use the Dirac sea as the fundamental description. Instead, it reinterprets the negative-frequency modes as creation operators for antiparticles. In the quantum field expansion, the same Dirac field contains electron annihilation operators and positron creation operators. This is one of the key conceptual moves from relativistic wave mechanics to quantum field theory (Weinberg 1995; Peskin and Schroeder 1995).

Example: what "negative frequency" becomes

A classical-looking Dirac solution contains terms like

$$u(p)e^{-ip\cdot x} \quad \text{and} \quad v(p)e^{+ip\cdot x},$$

with $p^0=+\sqrt{(p^2+m^2)}$.

In a one-particle wave equation, the second term looks like a negative-energy mode because

$$e^{+iEt-i\mathbf{p}\cdot\mathbf{x}}$$

has the opposite time dependence from a positive-energy wave.

In field theory, this term is not interpreted as a particle with energy -E. Instead, after quantization, its coefficient becomes an operator that creates an antiparticle with positive energy +E. Thus the energy spectrum can be bounded below.

This is not a cosmetic change. It is a change in what the equation means.

2.8 Antiparticles are required by relativistic quantum theory

An antiparticle is not merely a particle with opposite electric charge. More precisely, it is the particle associated with the charge-conjugate degrees of freedom of a relativistic quantum field. For the electron, the antiparticle is the positron. It has the same mass and spin as the electron, but opposite electric charge.

Relativistic quantum theory strongly suggests antiparticles because energy and momentum are connected by

$$E^2 = \mathbf{p}^2 + m^2,$$

which naturally gives two frequency branches. Field quantization reorganizes these branches into positive-energy particles and positive-energy antiparticles.

For the Dirac field in QED:

- electron states have charge -e,
- positron states have charge +e,
- both have mass m_e,
- both have spin 1/2.

The existence of the positron was one of the great confirmations of the relativistic electron theory (Anderson 1933). In modern language, the electron and positron are two kinds of excitation of one underlying Dirac field.

Example: pair annihilation

An electron and a positron can annihilate into photons:

$$e^- + e^+ \to \gamma + \gamma.$$

This process cannot be described in a fixed one-electron quantum mechanics, because the number and type of particles changes. It is natural in QED because the theory contains fields whose quanta can be created and destroyed.

Energy, momentum, angular momentum, and charge are still conserved. What is not conserved is the number of electrons alone.

2.9 Spin from relativistic wave equations

In nonrelativistic quantum mechanics, spin is often introduced as an additional internal label. For example, a spin-1/2 electron wavefunction may be written as a two-component Pauli spinor,

$$\psi(t,\mathbf{x}) = \begin{pmatrix} \psi_\uparrow(t,\mathbf{x}) \\ \psi_\downarrow(t,\mathbf{x}) \end{pmatrix}.$$

The two components represent spin-up and spin-down amplitudes relative to a chosen axis.

In the Dirac equation, spin is not an optional decoration. It is built into the Lorentz-covariant structure of the equation. The matrices γ^μ require ψ to have four components in 3+1 dimensions, and those components encode spin and particle-antiparticle structure.

At rest, a positive-energy electron has energy

$$E = m.$$

There are two independent spin states, often called spin-up and spin-down. Similarly, a positron has two independent spin states. Thus the four independent rest-frame solutions match the four physical possibilities:

$$\text{electron spin up}, \quad \text{electron spin down}, \quad \text{positron spin up}, \quad \text{positron spin down}$$

This counting is schematic but useful. The exact spinor basis depends on representation and normalization conventions, which we will develop later.

Spin and magnetic moment

The Dirac equation also explains why the electron couples to a magnetic field as a spin-1/2 particle. In the nonrelativistic limit with electromagnetic coupling, it leads to the Pauli Hamiltonian,

$$H_{\mathrm{Pauli}} = \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2 + q\Phi - \frac{q}{2m}\boldsymbol{\sigma} \cdot \mathbf{B},$$

up to convention-dependent signs for the charge q. Here:

- Φ is the scalar potential,
- A is the vector potential,
- B=∇×A is the magnetic field,
- boldsymbolσ are the Pauli matrices.

The spin coupling term has the form

$$-\boldsymbol{\mu} \cdot \mathbf{B},$$

where boldsymbolμ is the magnetic moment. At the level of the Dirac equation, the electron has gyromagnetic factor

$$g = 2.$$

QED loop corrections later modify this to

$$g = 2(1 + a_e),$$

where a_e is the anomalous magnetic moment. That precision story will return in Chapters 14 and 19.

2.10 Minimal electromagnetic coupling

To describe a charged relativistic particle in an electromagnetic background, one replaces the ordinary derivative with a gauge-covariant derivative. For a particle of charge q,

$$\partial_\mu \longrightarrow D_\mu = \partial_\mu + iqA_\mu,$$

where

$$A_\mu = (\Phi, -\mathbf{A})$$

in our metric convention.

The word gauge-covariant means that the derivative transforms in a controlled way under local phase changes of the wavefunction. If

$$\psi(x) \to e^{-iq\Lambda(x)}\psi(x),$$

and

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x),$$

then

$$D_\mu\psi \to e^{-iq\Lambda(x)} D_\mu\psi.$$

This is the mathematical seed of QED. A local phase convention should not change physical predictions, and the electromagnetic potential is precisely the compensating field that makes local phase invariance possible.

For the Dirac equation, minimal coupling gives

$$(i\gamma^\mu D_\mu - m)\psi = 0.$$

For the Klein-Gordon equation, it gives

$$(D_\mu D^\mu + m^2)\phi = 0.$$

At this stage, Aμ may be treated as a prescribed classical background. Full QED goes further: Aμ itself becomes a quantum field.

Example: an electron in a static electric potential

Suppose an electron moves in a static electrostatic potential Φ(x). The electromagnetic four-potential is

$$A^\mu = (\Phi, \mathbf{0}).$$

The Dirac equation becomes a relativistic wave equation with potential energy

$$q\Phi.$$

For an electron, q=-e, so the potential energy is

$$-e\Phi.$$

This reproduces the expected sign: an electron is attracted toward positive electrostatic potential.

But if the potential becomes strong enough, the distinction between electron states and positron states can become physically important. A one-particle interpretation becomes strained because the background field can supply enough energy to create particle-antiparticle pairs.

2.11 The Klein paradox as a warning sign

The Klein paradox is a famous relativistic scattering puzzle. In simple terms, when a relativistic particle scatters from a very high potential step, the Dirac equation can predict reflection and transmission behavior that seems impossible in a one-particle interpretation. In particular, for sufficiently strong potentials, the transmitted wave may correspond to states that would be interpreted as negative-energy solutions in the naive picture.

The resolution is not that the Dirac equation is algebraically wrong. The resolution is that the physical process is no longer a one-particle process. A strong external field can produce particle-antiparticle pairs. The extra reflected or transmitted flux is then understood as involving newly created particles, not as a violation of probability conservation (Bjorken and Drell 1964).

Example: strong potential and pair creation

Imagine a region where an external electric field does work of order

$$2m_e$$

over a short distance. In ordinary units, this corresponds to the electron-positron rest energy

$$2m_ec^2.$$

If enough energy is available, the field can create an electron-positron pair. A theory that insists there is exactly one electron forever has no place for this possibility.

This is the heart of the matter: special relativity allows energy to become mass. Quantum mechanics allows transitions between states. Together, they demand a framework in which particle number can change.

2.12 Why fixed-particle-number quantum mechanics fails

A fixed-particle-number theory assumes that the number of particles is set once and for all. For example, in ordinary one-particle quantum mechanics, the state space contains exactly one particle. In nonrelativistic many-body quantum mechanics, one may choose a Hilbert space for exactly N particles.

This works well when particle creation is energetically impossible or negligible. For example, an electron in a low-energy atomic orbital can often be treated nonrelativistically with a fixed number of particles. Pair creation effects are far too small to be part of the leading description.

But relativistic quantum mechanics must handle processes such as

$$e^- + e^+ \to \gamma + \gamma,$$

$$\gamma + \gamma \to e^- + e^+,$$

$$e^- \to e^- + \gamma$$

inside scattering amplitudes when allowed by energy-momentum conservation and by the presence of other particles or fields. The last process cannot occur for a free on-shell electron in vacuum because energy-momentum conservation forbids it, but virtual emission and absorption are central in interactions. In matter or external fields, real radiation is possible.

The deeper point is that relativistic interactions do not preserve the number of particles of each kind. They preserve charges and spacetime symmetries.

In QED, the conserved electric charge is

$$Q = (-e)N_{e^-} + (+e)N_{e^+},$$

not the electron number $N_{(e^-)}$ by itself.

For example, pair creation changes

$$N_{e^-} \to N_{e^-} + 1, \qquad N_{e^+} \to N_{e^+} + 1,$$

but the total charge changes by

$$(-e) + (+e) = 0.$$

So electric charge is conserved even though particle number changes.

This is exactly the kind of bookkeeping that quantum field theory handles naturally. A field has creation and annihilation operators. These operators can add or remove quanta while preserving the symmetries of the theory.

2.13 From wavefunctions to fields

The transition to quantum field theory can be summarized in one sentence:

> The object that looked like a relativistic wavefunction becomes an operator-valued field.

For the Dirac case, instead of treating ψ(x) as a one-electron wavefunction, QED treats ψ(x) as a quantum field operator. Roughly,

$$\psi(x) \sim \sum_s \int \frac{d^3p}{(2\pi)^3} \left[b_s(\mathbf{p})u_s(p)e^{-ip\cdot x} + d_s^\dagger(\mathbf{p})v_s(p)e^{+ip\cdot x}\right].$$

This formula will be derived carefully in Chapter 4. For now, read it conceptually:

- $b_s(p)$ annihilates an electron with momentum p and spin s,
- b_s^†(p) creates an electron,
- $d_s(p)$ annihilates a positron,
- d_s^†(p) creates a positron,
- $u_s(p)$ and $v_s(p)$ are spinor wavefunctions,
- all energies p^0 are taken positive in the particle interpretation.

The negative-frequency part of the classical solution becomes the antiparticle creation part of the quantum field.

For a complex scalar field, one similarly writes

$$\phi(x) \sim \int \frac{d^3p}{(2\pi)^3} \left[a(\mathbf{p})e^{-ip\cdot x} + b^\dagger(\mathbf{p})e^{+ip\cdot x}\right],$$

where a^† creates a particle and b^† creates an antiparticle.

This is the modern interpretation anticipated by the difficulties of relativistic wave mechanics and systematically developed in quantum field theory texts (Weinberg 1995; Peskin and Schroeder 1995).

2.14 What survives from relativistic quantum mechanics?

It would be a mistake to think that relativistic quantum mechanics is simply thrown away. Much of it survives, but its interpretation changes.

The Klein-Gordon equation survives as the free field equation for spin-0 fields:

$$(\Box + m^2)\phi = 0.$$

The Dirac equation survives as the free field equation for spin-1/2 fields:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0.$$

The conserved currents survive, but they are interpreted as charge and probability-related currents in the field-theoretic setting, not always as one-particle probability densities.

The plane-wave solutions survive as building blocks of field expansions.

The spinor algebra survives as the practical language of fermion scattering amplitudes.

The relativistic energy relation survives as the on-shell condition for external particles:

$$p^2 = m^2.$$

What does not survive is the idea that a relativistic interacting theory can generally be described using a Hilbert space with one fixed number of particles. QED requires Fock space, a Hilbert space built as a direct sum of sectors with different particle numbers. We will construct this explicitly in Chapter 4.

2.15 A first view of Fock space

A Hilbert space is the vector space of quantum states, equipped with an inner product that allows probabilities to be computed.

A Fock space is a Hilbert space designed for variable particle number. It has the schematic form

$$\mathcal{F} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots .$$

Here:

- $\mathcal{H}_0$ is the zero-particle sector, containing the vacuum state,
- $\mathcal{H}_1$ is the one-particle sector,
- $\mathcal{H}_2$ is the two-particle sector,

- and so on.

The symbol ⊕ means direct sum: a state may be a superposition involving different particle numbers.

For QED, a state might contain components such as

$$|e^-\rangle,$$

$$|e^-\gamma\rangle,$$

$$|e^-e^-e^+\rangle,$$

and many others, as long as conserved quantities such as total electric charge are respected.

For example, a physical electron in interacting QED is not merely a bare one-electron state. It is surrounded by quantum fluctuations: virtual photons, and at higher orders virtual electron-positron pairs. Perturbation theory organizes these contributions systematically.

This is why the later chapters spend so much time on creation operators, annihilation operators, propagators, and Feynman diagrams. They are not optional computational tricks. They are the natural language of a theory in which particles can be created and destroyed.

2.16 The conceptual map so far

Let us now connect the main ideas.

Special relativity gives

$$E^2 = \mathbf{p}^2 + m^2.$$

Quantizing this directly gives the Klein-Gordon equation,

$$(\Box + m^2)\phi = 0.$$

It is Lorentz invariant, but its conserved density is not positive definite if interpreted as probability. Field theory reinterprets its current as charge current.

Dirac sought a first-order relativistic equation and found

$$(i\gamma^\mu\partial_\mu - m)\psi = 0.$$

This equation has a positive density

$$\psi^\dagger\psi,$$

naturally incorporates spin-1/2, and correctly points toward the electron's magnetic properties. But it still contains negative-energy solutions.

The negative-energy problem is resolved by field quantization:

- positive-frequency modes annihilate particles,
- negative-frequency modes create antiparticles,
- all physical particles and antiparticles carry positive energy,
- particle number is allowed to change.

Thus relativistic quantum mechanics does not fail because its equations are useless. It fails because its one-particle interpretation is too small. The equations are trying to become field equations.

2.17 Debunk me: common misunderstandings

Before moving on, test the following claims carefully.

Claim 1: "The Klein-Gordon equation is wrong because its probability density can be negative."

Debunking: The Klein-Gordon equation is not wrong. The mistake is interpreting its conserved current as a one-particle probability current in the same way as the Schrödinger current. In quantum field theory, the Klein-Gordon field describes spin-0 particles, and the conserved current of a complex scalar field is interpreted as a charge current.

Claim 2: "The Dirac equation solved the negative-energy problem completely."

Debunking: The Dirac equation gives a positive density $\psi^\dagger\psi$, but it still has negative-energy solutions. The stable modern interpretation requires quantum fields and antiparticles.

Claim 3: “Antiparticles were added by hand.”

Debunking: Antiparticles are strongly suggested by the structure of relativistic wave equations and become unavoidable in local relativistic quantum field theory. The Dirac equation led historically to the positron idea, and Anderson observed the positron experimentally (Dirac 1931; Anderson 1933).

Claim 4: “QED violates particle-number conservation, so it violates conservation laws.”

Debunking: QED does not conserve the number of electrons or photons separately in general. It does conserve electric charge, energy, momentum, and angular momentum in processes where the corresponding symmetries apply.

Claim 5: “A field is just a wavefunction with another name.”

Debunking: A quantum field is an operator-valued object that can create and

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