Reading the Operator Exponential as a Power Series

The highlighted passage

$$e^{-iHt/\hbar} = I - \frac{iHt}{\hbar} + \frac{1}{2!}\left(-\frac{iHt}{\hbar}\right)^2 + \frac{1}{3!}\left(-\frac{iHt}{\hbar}\right)^3 + \cdots$$

spells out what is meant by exponentiating the Hamiltonian H. In ordinary calculus, the exponential function can be defined by its Taylor series,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

The highlighted formula is the same definition with the scalar variable x replaced by the operator

$$x = -\frac{iHt}{\hbar}.$$

That replacement is simple in appearance but conceptually important. The Hamiltonian H is not generally a number; it is a linear operator acting on quantum states. In a finite-dimensional quantum system, such as an ideal qubit, H can be represented by a matrix. The exponential e$^{(}$-iHt/hbar$^{)}$ is then a matrix exponential, and the powers in the series mean repeated matrix multiplication [Higham 2008; Nielsen and Chuang 2010].

Why the First Term Is I, Not 1

The ordinary exponential begins with 1, because 1 is the multiplicative identity for numbers. An operator exponential begins with I, because I is the identity operator:

$$I\left|\psi\right\rangle = \left|\psi\right\rangle .$$

This is the operator that leaves every state unchanged. Since the exponential is supposed to produce another operator, every term in its series must also be an operator. The first term is therefore I, the second term is -iHt/hbar, the third term is one half of the operator product (-iHt/hbar)(-iHt/hbar), and so on.

For example,

$$\left(-\frac{iHt}{\hbar}\right)^2 = \left(-\frac{it}{\hbar}\right)^2 H^2,$$

where

$$H^2 = HH.$$

This means: apply H, then apply H again. Similarly, H³=HHH. In finite dimensions this is just ordinary matrix multiplication. The factorials 2!,3!,… are the same factorials that appear in the scalar Taylor series.

One small but useful consistency check is dimensional. The exponential function must take a dimensionless argument. The Hamiltonian has units of energy, while t has units of time. The product Ht has units of energy times time, the same units as hbar. Thus

$$\frac{Ht}{\hbar}$$

is dimensionless. This is one reason the combination Ht/hbar appears naturally in Schrödinger time evolution.

How the Series Solves Schrödinger’s Equation

The parent document identifies

$$U(t) = e^{-iHt/\hbar}$$

as the time-evolution operator for a closed system with a time-independent Hamiltonian. The highlighted passage gives the internal meaning of that compact exponential notation. To see why this series is the right one, begin with the time-dependent Schrödinger equation,

$$i\hbar\frac{d}{dt}\left|\psi(t)\right\rangle = H\left|\psi(t)\right\rangle.$$

If the state is written as

$$|\psi(t)\rangle = U(t)\,|\psi(0)\rangle\,,$$

then U(t) itself must satisfy

$$i\hbar\frac{dU(t)}{dt} = HU(t), \qquad U(0) = I.$$

The initial condition U(0)=I says that at time zero, no evolution has yet occurred.

Now insert the series

$$U(t) = I - \frac{iHt}{\hbar} + \frac{1}{2!}\left(-\frac{iHt}{\hbar}\right)^2 + \cdots .$$

At t=0, every term containing t vanishes, leaving U(0)=I. Differentiating term by term gives

$$\frac{dU(t)}{dt} = -\frac{iH}{\hbar}\left[I - \frac{iHt}{\hbar} + \frac{1}{2!}\left(-\frac{iHt}{\hbar}\right)^2 + \cdots\right],$$

which is

$$\frac{dU(t)}{dt} = -\frac{iH}{\hbar}U(t).$$

Multiplying both sides by ihbar gives

$$i\hbar\frac{dU(t)}{dt} = HU(t).$$

So the power series is not merely a formal decoration. Under the usual finite-dimensional assumptions, it directly solves the evolution equation [Sakurai and Napolitano 2017].

What the Powers of H Do Physically

The series can look abstract until one examines an energy eigenstate. Suppose |E⟩ is an eigenstate of the Hamiltonian:

$$H\,|E\rangle = E\,|E\rangle\,,$$

where E is the corresponding energy eigenvalue. Applying the series to this state gives

$$e^{-iHt/\hbar}\,|E\rangle = \left[I - \frac{iHt}{\hbar} + \frac{1}{2!}\left(-\frac{iHt}{\hbar}\right)^2 + \cdots\right]|E\rangle\,.$$

Each power of H acts simply:

$$H^2\,|E\rangle = E^2\,|E\rangle\,, \qquad H^3\,|E\rangle = E^3\,|E\rangle\,,$$

and so forth. Therefore the operator series becomes

$$\left[1 - \frac{iEt}{\hbar} + \frac{1}{2!}\left(-\frac{iEt}{\hbar}\right)^2 + \frac{1}{3!}\left(-\frac{iEt}{\hbar}\right)^3 + \cdots\right]|E\rangle\,.$$

The bracket is now an ordinary scalar exponential:

$$e^{-iEt/\hbar}\,|E\rangle\,.$$

This is the bridge between the operator formula and the phase factor emphasized in the parent document. The parent document directly supports the interpretation that definite-energy components acquire phases of the form $e^{(-iEt/hbar)}$. The highlighted passage explains how that result follows from the operator exponential when expanded term by term.

For a superposition of energy eigenstates,

$$|\psi(0)\rangle = \sum_n c_n\,|E_n\rangle\,,$$

linearity gives

$$|\psi(t)\rangle = \sum_n c_n e^{-iE_n t/\hbar} |E_n\rangle \,.$$

Each energy component receives its own phase. Observable interference depends on relative phases such as

$$e^{-i(E_m - E_n)t/\hbar},$$

not usually on a single global phase common to the whole state.

A Qubit Example Hidden in the Series

For a two-level system, the Hamiltonian is often expressed using Pauli matrices. Consider

$$H = \frac{\hbar\omega}{2}\sigma_z,$$

where

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then

$$U(t) = e^{-i\omega t \sigma_z/2}.$$

The series becomes

$$I - \frac{i\omega t}{2}\sigma_z + \frac{1}{2!}\left(-\frac{i\omega t}{2}\right)^2 \sigma_z^2 + \frac{1}{3!}\left(-\frac{i\omega t}{2}\right)^3 \sigma_z^3 + \cdots \,.$$

Because

$$\sigma_z^2 = I, \qquad \sigma_z^3 = \sigma_z,$$

the even powers contribute multiples of I, while the odd powers contribute multiples of σ₍z₎. Grouping even and odd terms yields

$$e^{-i\omega t\sigma_z/2} = \cos\left(\frac{\omega t}{2}\right) I - i\sin\left(\frac{\omega t}{2}\right) \sigma_z.$$

This is a standard single-qubit rotation form [Nielsen and Chuang 2010]. It also shows why the exponential of an operator can often be simplified when the operator has a special algebraic property. Here the simple identity σ₍z₎²=I makes the infinite series collapse into familiar sine and cosine functions.

Convergence and the Boundary of the Formula

For finite-dimensional matrices, the power series for the exponential converges for every matrix and every finite value of t [Higham 2008]. This means the highlighted expansion is mathematically safe in the usual qubit and finite-dimensional quantum-computation setting.

The situation requires more care for infinite-dimensional Hilbert spaces. Many physically important Hamiltonians, such as the free-particle Hamiltonian or harmonic oscillator Hamiltonian, are unbounded operators. For such operators, the power series expression may not be meaningful on every vector in the Hilbert space. In rigorous quantum mechanics, e⁽-iHt/hbar⁾ is then usually defined through the spectral theorem for self-adjoint operators, not simply by applying the power series to all states. Stone’s theorem gives the deeper relationship: strongly continuous one-parameter unitary groups are generated by self-adjoint operators [Stone 1932; Reed and Simon 1980]. The highlighted series remains an excellent guide in finite dimensions and for many formal calculations, but its domain of validity should not be assumed without checking the operator setting.

There is also an important physical assumption behind the compact expression: H is time-independent. If the Hamiltonian depends on time, the evolution is not generally

$$e^{-\frac{i}{\hbar}\int H(t)\,dt}$$

unless Hamiltonians at different times commute. In the general time-dependent case, the correct object involves time ordering, often written as

$$U(t) = \mathcal{T} \exp\left(-\frac{i}{\hbar}\int_0^t H(t')\,dt'\right).$$

That is a related but more delicate expansion, usually developed as the Dyson series [Sakurai and Napolitano 2017].

The highlighted passage is therefore best read as the finite-dimensional or suitably well-behaved power-series definition of the time-evolution operator for a time-independent Hamiltonian. It supports the parent document’s movement from energy to phase, but it does not by itself verify a particular physical model. To verify an application, one should check that the Hamiltonian is the correct self-adjoint generator, that it is effectively time-independent or treated with time ordering when necessary, that the basis states used are truly energy eigenstates or are otherwise evolved correctly, and that the operator exponential is being interpreted in a mathematically valid domain.

References

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Document information

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