Chapter 1: Qubits, States, and Measurement

A quantum circuit is built from three ingredients: qubits, gates, and measurements. Gates and circuit diagrams will come next. In this chapter we study the object that travels along a quantum circuit wire: the qubit.

A classical bit has two possible values, usually written as 0 and 1. A qubit also has two standard labels, written $|0\rangle$ and $|1\rangle$, but its state is not limited to being only one of those two. Before measurement, the state of a single ideal qubit is represented by a vector in a two-dimensional complex vector space. This is one of the basic postulates of quantum mechanics as used in quantum computation (Nielsen and Chuang, 2010).

The goal of this chapter is to make that sentence feel natural:

> A qubit state is a normalized vector in mathbbC², usually written >

$$> |\psi\rangle = \alpha|0\rangle + \beta|1\rangle, >$$

> where α,β ∈ mathbbC and >

$$> |\alpha|^2 + |\beta|^2 = 1. >$$

We will unpack every part of this statement carefully.

1.1 From bits to qubits

A bit is the basic unit of classical information. It can be in one of two states:

$$0 \quad \text{or} \quad 1.$$

For example, a classical wire in a circuit may carry a voltage that represents 0 or a voltage that represents 1. A classical NOT gate flips the value:

$$0 \mapsto 1, \qquad 1 \mapsto 0.$$

A qubit, short for quantum bit, is the basic unit of quantum information. It also has two distinguished states:

$$|0\rangle \quad \text{and} \quad |1\rangle.$$

The notation $|\cdot\rangle$ is called ket notation or Dirac notation. It is a compact way to write vectors used throughout quantum mechanics and quantum computing (Nielsen and Chuang, 2010).

The symbols $|0\rangle$ and $|1\rangle$ are not ordinary numbers. They are vectors:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

These two vectors form the computational basis for one qubit. A basis is a set of vectors from which every vector in the space can be built uniquely by scaling and adding. In this case, every single-qubit state vector can be written as a combination of $|0\rangle$ and $|1\rangle$.

So a qubit can be in a state such as

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle.$$

This is not the same as saying the qubit is secretly either $|0\rangle$ or $|1\rangle$ and we simply do not know which. It means the state vector itself is a combination of both basis vectors. When we later measure it, we get one classical outcome, but before measurement the mathematical state is the vector above.

1.2 Complex numbers: the numbers used for amplitudes

The coefficients in a qubit state are usually complex numbers. A complex number has the form

$$z = a + bi,$$

where a and b are real numbers, and i is defined by

$$i^2 = -1.$$

For example,

$$3 + 2i, \qquad -1 + 5i, \qquad \frac{1}{\sqrt{2}}, \qquad -i$$

are complex numbers.

The complex number

$$z = a + bi$$

has a complex conjugate

$$z^* = a - bi.$$

Its squared magnitude is

$$|z|^2 = z^* z = a^2 + b^2.$$

For example, if

$$z = 3 + 4i,$$

then

$$|z|^2 = 3^2 + 4^2 = 25,$$

so

$$|z| = 5.$$

In quantum mechanics, the coefficients of a state vector are called amplitudes. Amplitudes are not probabilities themselves. Probabilities come from squared magnitudes of amplitudes. This rule is often called the Born rule in quantum mechanics (Griffiths and Schroeter, 2018).

For a one-qubit state

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle,$$

α is the amplitude of $|0\rangle$, and β is the amplitude of $|1\rangle$.

If we measure this qubit in the computational basis, then:

$$P(0) = |\alpha|^2, \qquad P(1) = |\beta|^2.$$

That is why complex numbers matter: their magnitudes give probabilities, and their phases affect interference.

1.3 The state vector of one qubit

A pure state of one ideal qubit is written as

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle,$$

where

$$\alpha, \beta \in \mathbb{C}.$$

Using column vectors, this is

$$|\psi\rangle = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

So the expression

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

and the column vector

$$|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

mean the same thing.

The state must be normalized, meaning its total probability must equal 1:

$$|\alpha|^2 + |\beta|^2 = 1.$$

This condition is essential. If measuring the qubit can produce outcome 0 or outcome 1, then the probabilities of all possible outcomes must add to 1.

Example: a valid qubit state

Consider

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle.$$

Here,

$$\alpha = \frac{1}{\sqrt{2}}, \qquad \beta = \frac{1}{\sqrt{2}}.$$

Check normalization:

$$|\alpha|^2 + |\beta|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 + \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2} + \frac{1}{2} = 1.$$

So this is a valid qubit state.

If we measure it in the computational basis, then

$$P(0) = \frac{1}{2}, \qquad P(1) = \frac{1}{2}.$$

Example: an invalid unnormalized state

Now consider

$$|\phi\rangle = |0\rangle + |1\rangle.$$

This looks similar, but its amplitudes are

$$\alpha = 1, \qquad \beta = 1.$$

Then

$$|\alpha|^2 + |\beta|^2 = 1 + 1 = 2.$$

This is not normalized, so it is not a valid physical state as written.

However, we can normalize it. To normalize a nonzero vector, divide it by its length. The length of

$$|0\rangle + |1\rangle$$

is

$$\sqrt{2}.$$

So the normalized version is

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle.$$

This normalized state is valid.

1.4 Superposition

A superposition is a linear combination of basis states. For one qubit,

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

is a superposition of $|0\rangle$ and $|1\rangle$ when both amplitudes may contribute.

The word “superposition” can sound mysterious, but mathematically it means something simple: scaled vectors added together.

For example,

$$|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$$

is a superposition. It is normalized because

$$\left|\frac{\sqrt{3}}{2}\right|^2 + \left|\frac{1}{2}\right|^2 = \frac{3}{4} + \frac{1}{4} = 1.$$

If measured in the computational basis,

$$P(0) = \frac{3}{4}, \qquad P(1) = \frac{1}{4}.$$

So this qubit is more likely to produce 0 than 1, but the state before measurement is not simply the classical probability distribution “75% zero, 25% one.” The amplitudes can be complex, and their phases can later interfere when gates are applied. This difference between amplitudes and ordinary probabilities is one of the central reasons quantum circuits behave differently from classical randomized circuits (Mermin, 2007).

1.5 Inner products and orthogonality

To reason about quantum states, we need a way to compare vectors. The main tool is the inner product.

For two one-qubit states

$$|\psi\rangle = \begin{bmatrix}\alpha \\ \beta\end{bmatrix}, \qquad |\phi\rangle = \begin{bmatrix}\gamma \\ \delta\end{bmatrix},$$

their inner product is

$$\langle\psi|\phi\rangle = \alpha^*\gamma + \beta^*\delta.$$

The notation ⟨ ψ| is called a bra. It is the conjugate transpose of the ket |ψ⟩. If

$$|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix},$$

then

$$\langle\psi| = \begin{bmatrix} \alpha^* & \beta^* \end{bmatrix}.$$

So a ket is a column vector, and a bra is the corresponding conjugate row vector.

The norm, or length, of a state is computed by

$$\langle\psi|\psi\rangle.$$

For

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle,$$

we have

$$\langle\psi|\psi\rangle = |\alpha|^2 + |\beta|^2.$$

A normalized state satisfies

$$\langle\psi|\psi\rangle = 1.$$

Two states are orthogonal if their inner product is zero. Orthogonal states are perfectly distinguishable by an appropriate measurement in quantum mechanics (Nielsen and Chuang, 2010).

For example,

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Their inner product is

$$\langle 0|1\rangle = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0.$$

So |0⟩ and |1⟩ are orthogonal.

Now define two new states:

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \qquad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

These are also orthogonal:

$$\langle +|-\rangle = \left(\frac{\langle 0| + \langle 1|}{\sqrt{2}} \right) \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right).$$

Expanding,

$$\langle +|-\rangle = \frac{1}{2} \left(\langle 0|0\rangle - \langle 0|1\rangle + \langle 1|0\rangle - \langle 1|1\rangle \right).$$

Since

$$\langle 0|0\rangle = 1, \quad \langle 0|1\rangle = 0, \quad \langle 1|0\rangle = 0, \quad \langle 1|1\rangle = 1,$$

we get

$$\langle +|-\rangle = \frac{1}{2}(1 - 0 + 0 - 1) = 0.$$

Thus |+⟩ and |-⟩ form another valid basis for a single qubit.

1.6 Measurement in the computational basis

A quantum measurement converts a quantum state into a classical outcome. In the simplest case, we measure a qubit in the computational basis:

$$\{|0\rangle, |1\rangle\}.$$

Suppose

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle.$$

Then measurement in the computational basis gives:

$$0 \quad \text{with probability} \quad |\alpha|^2,$$

and

$$1 \quad \text{with probability} \quad |\beta|^2.$$

After measurement, the state becomes the basis state corresponding to the observed outcome. If the outcome is 0, the post-measurement state is $|0\rangle$. If the outcome is 1, the post-measurement state is $|1\rangle$. This is the standard projective measurement rule used in introductory quantum computation (Nielsen and Chuang, 2010).

Example: measuring a balanced superposition

Let

$$|\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}.$$

Then

$$\alpha = \frac{1}{\sqrt{2}}, \qquad \beta = \frac{1}{\sqrt{2}}.$$

So

$$P(0) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2},$$

and

$$P(1) = \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2}.$$

A measurement gives either 0 or 1. It does not return “both.” If the result is 0, then immediately after measurement the state is

$$|0\rangle.$$

If the result is 1, then immediately after measurement the state is

$$|1\rangle.$$

If we prepare the same state many times and measure each copy once, we expect approximately half the outcomes to be 0 and half to be 1. For example, after 1000 repeated preparations and measurements, we might see a histogram like:

$$0 : 493, \qquad 1 : 507.$$

The exact counts vary from run to run because quantum measurement is probabilistic.

Example: measuring a biased superposition

Let

$$|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle.$$

Then

$$P(0) = \left| \frac{\sqrt{3}}{2} \right|^2 = \frac{3}{4},$$

and

$$P(1) = \left|\frac{1}{2}\right|^2 = \frac{1}{4}.$$

So if we prepare and measure this state many times, roughly 75\% of the results should be 0, and roughly 25\% should be 1.

1.7 Amplitudes are not probabilities

It is tempting to think of

$$\alpha|0\rangle + \beta|1\rangle$$

as a classical random mixture: “$|0\rangle$ with probability α, $|1\rangle$ with probability β.” That is not correct.

First, probabilities must be real numbers between 0 and 1, but amplitudes can be negative or complex. For example,

$$|\psi\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

has amplitudes

$$\alpha = \frac{1}{\sqrt{2}}, \qquad \beta = -\frac{1}{\sqrt{2}}.$$

The amplitude of $|1\rangle$ is negative, but the probability of measuring 1 is still positive:

$$P(1) = \left|-\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}.$$

Second, two states can produce the same measurement probabilities in one basis but behave differently under later operations or under measurement in another basis. This is where phases become important.

Consider

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}},$$

and

$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

If we measure either state in the computational basis, we get

$$P(0) = \frac{1}{2}, \qquad P(1) = \frac{1}{2}.$$

So computational-basis measurement alone cannot distinguish them.

But |+⟩ and |-⟩ are orthogonal states. If we measure in the basis

$$\{|+\rangle, |-\rangle\},$$

then |+⟩ is detected as |+⟩ with probability 1, while |-⟩ is detected as |-⟩ with probability 1.

Thus the sign difference matters. It is not visible in the computational-basis probabilities alone, but it is physically meaningful.

1.8 Global phase and relative phase

A complex number can be written in polar form:

$$re^{i\theta},$$

where r ≥ 0, θ is an angle, and

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

This identity is Euler's formula.

In quantum states, phases appear in amplitudes. For example,

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

and

$$\frac{|0\rangle + i|1\rangle}{\sqrt{2}}$$

have the same computational-basis probabilities:

$$P(0) = \frac{1}{2}, \qquad P(1) = \frac{1}{2}.$$

But they are not the same physical state, because the relative phase between the $|0\rangle$ and $|1\rangle$ components is different.

There are two important kinds of phase:

1. Global phase
2. Relative phase

A global phase multiplies the entire state by the same complex phase factor:

$$|\psi\rangle \mapsto e^{i\gamma}|\psi\rangle.$$

For example,

$$|\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}},$$

and

$$|\phi\rangle = -i\frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

differ only by the global phase -i. These two vectors represent the same physical pure state. Global phase does not affect measurement probabilities or any observable experimental prediction (Nielsen and Chuang, 2010).

To see why, suppose

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle.$$

Now multiply the whole state by e⁽iγ⁾:

$$|\phi\rangle = e^{i\gamma}\alpha|0\rangle + e^{i\gamma}\beta|1\rangle.$$

The probability of measuring 0 is

$$|e^{i\gamma}\alpha|^2.$$

But

$$|e^{i\gamma}|^2 = 1,$$

so

$$|e^{i\gamma}\alpha|^2 = |\alpha|^2.$$

Similarly,

$$|e^{i\gamma}\beta|^2 = |\beta|^2.$$

Thus computational-basis probabilities do not change. More generally, all measurement predictions remain unchanged.

A relative phase changes the phase of one component compared with another. For example,

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

and

$$\frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

do not differ by a global phase. The minus sign changes the relative phase between the $|0\rangle$ and $|1\rangle$ terms. Relative phase is physically meaningful because it can affect interference.

Example: same probabilities, different state

Compare

$$|\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = |+\rangle$$

with

$$|\phi\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} = |-\rangle.$$

Both give

$$P(0) = P(1) = \frac{1}{2}$$

when measured in the computational basis.

But their inner product is

$$\langle + | - \rangle = 0.$$

So they are orthogonal. They are as different as $|0\rangle$ and $|1\rangle$, just in a different basis.

This is the first hint of quantum interference. In later chapters, gates will transform relative phases into observable differences in measurement probabilities.

1.9 Measurement in another basis

The computational basis is not the only possible measurement basis. Any pair of orthonormal one-qubit states can serve as a measurement basis.

An orthonormal basis is a set of vectors that satisfies two conditions:

1. Each vector has length 1.
2. Different vectors have inner product 0.

For example,

$$\{|0\rangle, |1\rangle\}$$

is an orthonormal basis.

Also,

$$\{|+\rangle, |-\rangle\}$$

is an orthonormal basis.

Suppose a qubit is in state |ψ⟩, and we measure in an orthonormal basis

$$\{|u\rangle, |v\rangle\}.$$

Then the probability of outcome u is

$$|\langle u|\psi\rangle|^2,$$

and the probability of outcome v is

$$|\langle v|\psi\rangle|^2.$$

This is the same Born rule, expressed using inner products (Griffiths and Schroeter, 2018).

Example: measuring |0⟩ in the \|+⟩, |-⟩\ basis

Let

$$|\psi\rangle = |0\rangle.$$

We measure in the basis

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \qquad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

Compute the probability of outcome +:

$$P(+) = |\langle +|0\rangle|^2.$$

Now

$$\langle +| = \frac{\langle 0| + \langle 1|}{\sqrt{2}}.$$

So

$$\langle +|0\rangle = \frac{\langle 0|0\rangle + \langle 1|0\rangle}{\sqrt{2}} = \frac{1+0}{\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

Thus

$$P(+) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}.$$

Similarly,

$$P(-) = |\langle -|0\rangle|^2 = \frac{1}{2}.$$

So if $|0\rangle$ is measured in the plus-minus basis, the outcomes + and - are equally likely.

This example teaches an important lesson: measurement probabilities depend on both the state and the measurement basis.

1.10 The Bloch sphere

A single-qubit pure state has two complex amplitudes:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle.$$

At first, this seems like many numbers. Since α and β are complex, we might think we need four real numbers. But normalization removes one degree of freedom, and global phase does not change the physical state. What remains can be represented by two real angles.

Every single-qubit pure state can be written, up to global phase, as

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle,$$

where

$$0 \le \theta \le \pi, \qquad 0 \le \varphi < 2\pi.$$

This representation is the basis for the Bloch sphere, a geometric picture of one-qubit pure states (Nielsen and Chuang, 2010).

The corresponding point on the sphere has coordinates

$$x = \sin\theta\cos\varphi,$$

$$y = \sin\theta\sin\varphi,$$

$$z = \cos\theta.$$

So each pure qubit state corresponds to a point on the surface of a unit sphere, after ignoring global phase.

Important examples:

$$|0\rangle$$

is at the north pole:

$$\theta = 0.$$

$$|1\rangle$$

is at the south pole:

$$\theta = \pi.$$

The state

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

has

$$\theta = \frac{\pi}{2}, \qquad \varphi = 0.$$

It lies on the positive x-axis.

The state

$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

has

$$\theta = \frac{\pi}{2}, \qquad \varphi = \pi.$$

It lies on the negative x-axis.

The state

$$|+i\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$$

has

$$\theta = \frac{\pi}{2}, \qquad \varphi = \frac{\pi}{2}.$$

It lies on the positive y-axis.

The Bloch sphere is especially useful for visualizing single-qubit gates, which we study in the next chapter. Many single-qubit gates can be understood as rotations of the Bloch sphere.

However, the Bloch sphere is mainly a one-qubit visualization tool. For two or more qubits, the state space grows much larger, and the geometry is no longer captured by an ordinary sphere. We will study multi-qubit states in Chapter 3.

1.11 A careful warning: superposition is basis-dependent

A state may look like a superposition in one basis but not in another.

For example,

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

is a superposition when written in the computational basis.

But in the plus-minus basis

$$\{|+\rangle, |-\rangle\},$$

the same state is simply the first basis vector:

$$|+\rangle = 1|+\rangle + 0|-\rangle.$$

So whether a state is “a superposition” depends on which basis you are using. This is not a contradiction. A vector can be simple in one coordinate system and a combination in another.

A familiar analogy is a two-dimensional arrow in the plane. In the usual x,y coordinates, an arrow may have both x- and y-components. But if you rotate the coordinate axes so that one axis points exactly along the arrow, then the same arrow has only one nonzero coordinate.

Quantum states behave similarly, except the vector space is complex and measurement probabilities come from squared magnitudes of amplitudes.

1.12 What measurement does and does not say

Measurement is one of the easiest parts of quantum computing to calculate, but one of the easiest parts to misunderstand.

Suppose

$$|\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}.$$

When measured in the computational basis, this state gives 0 with probability 1/2 and 1 with probability 1/2.

This does not mean that a single measurement outputs the probability distribution. A single measurement gives one classical result:

$$0 \quad \text{or} \quad 1.$$

To estimate the probabilities experimentally, we must prepare the same state many times and measure each copy. Quantum computers and simulators often call these repeated runs shots. If we run 1000 shots, we get 1000 classical outcomes. From those outcomes we estimate probabilities.

For example, if the true probabilities are

$$P(0) = \frac{1}{2}, \qquad P(1) = \frac{1}{2},$$

then 1000 shots might produce

$$0 : 512, \qquad 1 : 488.$$

Another run might produce

$$0 : 497, \qquad 1 : 503.$$

Both are consistent with the same underlying quantum state.

A statevector simulator may show the exact amplitudes, such as

$$\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

Real hardware does not directly print the statevector. It returns measurement outcomes, affected by both quantum probabilities and physical imperfections. We return to this distinction in Chapter 8.

1.13 How to read a one-qubit state

When you see a one-qubit state, read it in four steps.

Suppose

$$|\psi\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}i|1\rangle.$$

First, identify the amplitudes:

$$\alpha = \frac{1}{2}, \qquad \beta = \frac{\sqrt{3}}{2}i.$$

Second, check normalization:

$$|\alpha|^2 = \left|\frac{1}{2}\right|^2 = \frac{1}{4}.$$

Also,

$$|\beta|^2 = \left|\frac{\sqrt{3}}{2}i\right|^2 = \frac{3}{4}|i|^2.$$

Since

$$|i|^2 = 1,$$

we get

$$|\beta|^2 = \frac{3}{4}.$$

Thus

$$|\alpha|^2 + |\beta|^2 = \frac{1}{4} + \frac{3}{4} = 1.$$

So the state is normalized.

Third, compute computational-basis measurement probabilities:

$$P(0) = \frac{1}{4}, \qquad P(1) = \frac{3}{4}.$$

Fourth, notice the relative phase. The amplitude of $|1\rangle$ has a factor of i, so the state is not the same as

$$\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle.$$

Those two states have the same computational-basis probabilities, but they can behave differently under later gates.

This four-step habit will be useful throughout the book:

1. Identify amplitudes.
2. Check normalization.
3. Compute measurement probabilities.

4. Notice phases.

1.14 Mini-summary

A qubit is described by a normalized vector in a two-dimensional complex vector space:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle.$$

The basis states are

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The amplitudes α and β are complex numbers. They must satisfy

$$|\alpha|^2 + |\beta|^2 = 1.$$

Measurement in the computational basis gives

$$P(0) = |\alpha|^2, \qquad P(1) = |\beta|^2.$$

A global phase does not change the physical state:

$$|\psi\rangle \quad \text{and} \quad e^{i\gamma}|\psi\rangle$$

represent the same pure state.

A relative phase does matter. For example,

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

and

$$\frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

are different states, even though they give the same computational-basis measurement probabilities.

The Bloch sphere represents one-qubit pure states geometrically:

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle.$$

In the next chapter, we will learn how quantum gates transform these state vectors. That is the next step toward reading and building quantum circuits.

Exercises

Exercise 1: Check normalization

Which of the following are valid normalized one-qubit states?

1.

$$\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

2.

$$\frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle$$

3.

$$\frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle$$

4.

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