

Chapter 3: Hilbert Spaces and Operators

In Chapter 2 we built the linear-algebra toolkit: vectors, inner products, matrices, adjoints, eigenvalues, projections, positivity, and tensor products. In this chapter we reorganize that toolkit into the language used throughout quantum theory: Hilbert spaces and operators.

The change is partly conceptual. Instead of thinking only about column vectors and matrices, we begin to think about spaces of possible quantum state vectors and the transformations acting on them. This language is flexible enough for finite-dimensional quantum information and also points toward the infinite-dimensional theory used in mathematical physics.

For most of this book, our Hilbert spaces are finite-dimensional complex Hilbert spaces. This is the standard setting for qubits, finite quantum registers, and finite-outcome measurements in quantum information theory (Nielsen and Chuang, 2010; Watrous, 2018). Still, the word “Hilbert” carries an important idea: it is not just a vector space, but a vector space equipped with geometry.

The central objects of this chapter are:

- Hilbert spaces,
- bounded linear operators,
- adjoints,
- self-adjoint operators,
- positive semidefinite operators,
- isometries,
- partial isometries,
- direct sums.

These are exactly the tools we need before discussing projection-valued measurements, POVMs, and Naimark dilation.

3.1 From vector spaces to Hilbert spaces

A complex vector space is a space whose vectors can be added and multiplied by complex numbers. But quantum theory needs more than addition and scalar multiplication. It also needs a way to speak about lengths, angles, orthogonality, and probabilities.

That extra structure comes from an inner product.

A complex inner product on a vector space \mathcal{H} is a rule that assigns to every pair of vectors $\phi, \psi \in \mathcal{H}$ a complex number

$$\langle \phi, \psi \rangle$$

satisfying the following properties:

1. Conjugate symmetry:

$$\langle \phi, \psi \rangle = \overline{\langle \psi, \phi \rangle}.$$

2. Linearity in the second slot:

$$\langle \phi, a\psi_1 + b\psi_2 \rangle = a\langle \phi, \psi_1 \rangle + b\langle \phi, \psi_2 \rangle.$$

3. Conjugate linearity in the first slot:

$$\langle a\phi_1 + b\phi_2, \psi \rangle = \bar{a}\langle \phi_1, \psi \rangle + \bar{b}\langle \phi_2, \psi \rangle.$$

4. Positive definiteness:

$$\langle \psi, \psi \rangle \geq 0,$$

and

$$\langle \psi, \psi \rangle = 0 \quad \text{if and only if} \quad \psi = 0.$$

Some mathematics books use the opposite convention, making the inner product linear in the first slot. In quantum information, the convention above is common because of Dirac notation: $\langle \phi | \psi \rangle$ is linear in $|\psi\rangle$ and conjugate-linear in $\langle \phi|$. The physical predictions are independent of this convention as long as one is consistent (Nielsen and Chuang, 2010; Watrous, 2018).

The inner product gives a norm, or length, by

$$\|\psi\| = \sqrt{\langle \psi, \psi \rangle}.$$

A vector ψ is called a unit vector if

$$\|\psi\| = 1.$$

In quantum theory, pure states are represented by unit vectors, up to an overall complex phase. We will study that carefully in Chapter 4.

Example: \mathbb{C}^2 as a Hilbert space

The vector space \mathbb{C}^2 becomes a Hilbert space with the standard inner product

$$\left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \right\rangle = \bar{a}c + \bar{b}d.$$

For example, let

$$\psi = \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix}.$$

Then

$$\|\psi\|^2 = \left| \frac{1}{\sqrt{2}} \right|^2 + \left| \frac{i}{\sqrt{2}} \right|^2 = \frac{1}{2} + \frac{1}{2} = 1.$$

So ψ is a unit vector.

3.2 What makes a Hilbert space “Hilbert”?

An inner product space is a vector space with an inner product. A Hilbert space is an inner product space that is complete with respect to the norm induced by the inner product.

Let us unpack the word complete.

A sequence of vectors

$$\psi_1, \psi_2, \psi_3, \dots$$

is called a Cauchy sequence if its terms eventually become arbitrarily close to each other. Formally, for every $\varepsilon > 0$, there exists an integer N such that whenever $m, n \geq N$,

$$\|\psi_m - \psi_n\| < \varepsilon.$$

A normed space is complete if every Cauchy sequence converges to a vector inside the space.

So a Hilbert space is an inner product space with no “missing limit points.” This completeness condition is essential in infinite-dimensional analysis, where limits of sequences are unavoidable. Standard Hilbert-space theory is one of the mathematical foundations of quantum mechanics and functional analysis (Conway, 1990).

In finite dimensions, however, life is simpler:

> Every finite-dimensional inner product space over \mathbb{C} is complete.

Therefore, in most of this book, when we say “Hilbert space,” you may safely think:

> a finite-dimensional complex vector space with an inner product.

Still, remembering the completeness condition helps us understand why the same language also works in infinite-dimensional quantum theory.

3.3 Orthogonality and orthonormal bases

Two vectors $\phi, \psi \in \mathcal{H}$ are orthogonal if

$$\langle \phi, \psi \rangle = 0.$$

Orthogonality means that the vectors point in completely distinguishable directions inside the Hilbert-space geometry.

A list of vectors

$$e_1, e_2, \dots, e_n$$

is called orthonormal if

$$\langle e_i, e_j \rangle = \delta_{ij},$$

where

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

An orthonormal basis is an orthonormal list that spans the whole space.

If \mathbb{H} is n -dimensional and e_1, \dots, e_n is an orthonormal basis, then every vector $\psi \in \mathbb{H}$ can be written uniquely as

$$\psi = \sum_{j=1}^n c_j e_j,$$

where

$$c_j = \langle e_j, \psi \rangle.$$

Thus

$$\psi = \sum_{j=1}^n \langle e_j, \psi \rangle e_j.$$

This formula is one of the most important computational tools in quantum theory. It says that the components of a vector in an orthonormal basis are inner products with the basis vectors.

Example: the computational basis of a qubit

The Hilbert space of one qubit is \mathbb{C}^2 . Its standard orthonormal basis is

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Every qubit vector can be written as

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

for complex numbers α, β . If $|\psi\rangle$ is a unit vector, then

$$|\alpha|^2 + |\beta|^2 = 1.$$

Later, when we study projective measurements, the numbers $|\alpha|^2$ and $|\beta|^2$ will become probabilities.

3.4 Operators: transformations of Hilbert spaces

A linear operator from a Hilbert space \mathcal{H} to a Hilbert space \mathcal{K} is a function

$$A : \mathcal{H} \rightarrow \mathcal{K}$$

such that

$$A(a\psi + b\phi) = aA\psi + bA\phi$$

for all vectors $\psi, \phi \in \mathcal{H}$ and scalars $a, b \in \mathbb{C}$.

If $\mathcal{H} = \mathcal{K}$, we say that A is an operator on \mathcal{H} .

In finite dimensions, after choosing orthonormal bases, every linear operator is represented by a matrix. But the abstract operator notation is often clearer because many statements do not depend on a particular basis.

Example: a qubit bit-flip operator

Define $X: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ by

$$X|0\rangle = |1\rangle, \quad X|1\rangle = |0\rangle.$$

In the computational basis,

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

For a general qubit vector

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle,$$

we have

$$X|\psi\rangle = \alpha|1\rangle + \beta|0\rangle = \beta|0\rangle + \alpha|1\rangle.$$

So X swaps the two computational-basis amplitudes.

3.5 Bounded operators and the operator norm

A linear operator $A: \mathbb{C}^n \rightarrow \mathbb{C}^n$ is called bounded if there exists a constant $C \geq 0$ such that

$$\|A\psi\| \leq C\|\psi\|$$

for every $\psi \in \mathbb{C}^n$.

This means that A cannot stretch vectors by an unlimited factor. The smallest such constant is called the operator norm of A , written

$$\|A\|.$$

Equivalently,

$$\|A\| = \sup_{\|\psi\|=1} \|A\psi\|.$$

The symbol \sup means least upper bound. In finite dimensions, this supremum is actually attained, so it is a maximum.

In finite-dimensional Hilbert spaces, every linear operator is bounded. In infinite-dimensional Hilbert spaces, boundedness becomes a serious issue, and many important quantum observables, such as the idealized position and momentum operators, are unbounded operators with domain subtleties. This book mostly avoids those technical complications by working finite-dimensionally, as is common in quantum information theory (Nielsen and Chuang, 2010; Watrous, 2018). For the general functional-analytic setting, bounded operators are a central object of Hilbert-space theory (Conway, 1990).

Example: operator norm of a diagonal matrix

Let

$$A = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}.$$

For a unit vector

$$\psi = \begin{pmatrix} a \\ b \end{pmatrix}, \quad |a|^2 + |b|^2 = 1,$$

we get

$$A\psi = \begin{pmatrix} 2a \\ -3b \end{pmatrix}.$$

Thus

$$\|A\psi\|^2 = 4|a|^2 + 9|b|^2.$$

Because $|a|^2 + |b|^2 = 1$, this quantity is at most 9. Therefore

$$\|A\psi\| \leq 3$$

for every unit vector ψ , and equality occurs when $\psi = (0, 1)^T$. Hence

$$\|A\| = 3.$$

For a diagonal matrix, the operator norm is the largest absolute value of its diagonal entries.

3.6 The adjoint of an operator

The adjoint of an operator is the Hilbert-space version of conjugate transpose.

Let

$$A : \mathcal{H} \rightarrow \mathcal{K}$$

be a linear operator between finite-dimensional Hilbert spaces. The adjoint of A , denoted

$$A^* : \mathcal{K} \rightarrow \mathcal{H},$$

is the unique operator satisfying

$$\langle \phi, A\psi \rangle_{\mathcal{K}} = \langle A^*\phi, \psi \rangle_{\mathcal{H}}$$

for all $\psi \in \mathcal{H}$ and $\phi \in \mathcal{K}$.

This definition says: moving A from the second slot of the inner product to the first slot changes it into A^* .

In matrix form, if A is represented by a matrix, then A^* is represented by the conjugate transpose matrix:

$$A^* = \overline{A}^T.$$

That is, transpose the matrix and complex-conjugate each entry.

Example: computing an adjoint

Let

$$A = \begin{pmatrix} 1 & i \\ 2 & 0 \end{pmatrix}.$$

Then

$$A^* = \begin{pmatrix} 1 & 2 \\ -i & 0 \end{pmatrix}.$$

The entry i becomes $-i$, and rows become columns.

The adjoint obeys useful algebraic rules:

$$(A + B)^* = A^* + B^*,$$

$$(\lambda A)^* = \bar{\lambda} A^*,$$

$$(AB)^* = B^* A^*,$$

and

$$(A^*)^* = A.$$

The reversal in

$$(AB)^* = B^* A^*$$

is especially important. If AB means “first apply B , then apply A ,” then its adjoint reverses that order.

3.7 Self-adjoint operators

An operator $A: \square \rightarrow \square$ is called self-adjoint if

$$A = A^*.$$

In matrix language, self-adjoint matrices are also called Hermitian matrices.

Self-adjoint operators are central in quantum theory. In the traditional formulation of quantum mechanics, observables are represented by self-adjoint operators, and their spectral decompositions determine projective measurements (von Neumann, 1955; Nielsen and Chuang, 2010). In quantum information, we will also use self-adjoint operators to describe measurement effects, density matrices, and positive semidefinite operators.

A key property is:

> The eigenvalues of a finite-dimensional self-adjoint operator are real.

Here is the short proof. Suppose

$$A\psi = \lambda\psi$$

for some nonzero vector ψ . Then

$$\langle \psi, A\psi \rangle = \langle \psi, \lambda\psi \rangle = \lambda \langle \psi, \psi \rangle.$$

Since $A=A^*$,

$$\langle \psi, A\psi \rangle = \langle A\psi, \psi \rangle = \langle \lambda\psi, \psi \rangle = \bar{\lambda} \langle \psi, \psi \rangle.$$

Therefore

$$\lambda \langle \psi, \psi \rangle = \bar{\lambda} \langle \psi, \psi \rangle.$$

Because $\psi \neq 0$, we have $\langle \psi, \psi \rangle > 0$. Hence

$$\lambda = \bar{\lambda},$$

so λ is real.

Example: a self-adjoint qubit operator

The Pauli Z operator is

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since $Z^{\wedge}=Z$, it is self-adjoint. Its eigenvalues are 1 and -1, both real.

The Pauli Y operator is

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Although it contains imaginary entries, it is also self-adjoint because

$$Y^* = Y.$$

Self-adjoint does not mean “real matrix.” It means equal to its conjugate transpose.

3.8 Projections

A projection is an operator that represents projecting vectors onto a subspace.

In Hilbert-space theory, an orthogonal projection is an operator $P:\square\rightarrow\square$ satisfying

$$P^2 = P$$

and

$$P^* = P.$$

The equation $P^2=P$ says that applying P twice is the same as applying it once. Once a vector has been projected onto the subspace, projecting again changes nothing.

The equation $P^{\wedge}=P$ says that the projection is orthogonal rather than slanted.

Example: projection onto $|0\rangle$

In \mathbb{C}^2 , define

$$P_0 = |0\rangle\langle 0|.$$

In matrix form,

$$P_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

For

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle,$$

we have

$$P_0|\psi\rangle = \alpha|0\rangle.$$

So P_0 keeps the $|0\rangle$ component and removes the $|1\rangle$ component.

Check the projection equations:

$$P_0^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = P_0,$$

and

$$P_0^* = P_0.$$

Thus P_0 is an orthogonal projection.

Orthogonal projections will become the mathematical objects behind projective measurements in Chapter 5.

3.9 Positive semidefinite operators

A self-adjoint operator $A: \mathcal{H} \rightarrow \mathcal{H}$ is called positive semidefinite if

$$\langle \psi, A\psi \rangle \geq 0$$

for every $\psi \in \mathcal{H}$.

We write

$$A \geq 0.$$

This notation does not mean that every matrix entry of A is nonnegative. It means that every quadratic expression

$$\langle \psi, A\psi \rangle$$

is nonnegative.

Positive semidefinite operators are central to quantum information. Density matrices are positive semidefinite operators with trace 1. POVM elements are positive semidefinite operators that sum to the identity. These facts are standard in the mathematical formulation of quantum information theory (Nielsen and Chuang, 2010; Watrous, 2018).

Example: a positive semidefinite matrix

Let

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}.$$

For

$$\psi = \begin{pmatrix} a \\ b \end{pmatrix},$$

we have

$$\langle \psi, A\psi \rangle = 2|a|^2 + 5|b|^2 \geq 0.$$

So $A \geq 0$.

Example: positive semidefinite does not mean entrywise nonnegative

Consider

$$B = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

This matrix has negative entries, but for

$$\psi = \begin{pmatrix} a \\ b \end{pmatrix},$$

we compute

$$\langle \psi, B\psi \rangle = |a|^2 - \bar{a}b - \bar{b}a + |b|^2 = |a - b|^2 \geq 0.$$

Thus $B \geq 0$, even though some entries of B are negative.

Equivalent tests for positivity

In finite dimensions, for a self-adjoint operator A , the following conditions are equivalent:

1. $A \geq 0$.
2. All eigenvalues of A are nonnegative.
3. There exists an operator B such that

$$A = B^*B.$$

4. There exists a unique positive semidefinite operator $A^{1/2}$ such that

$$A^{1/2}A^{1/2} = A.$$

The operator $A^{1/2}$ is called the positive square root of A .

The positive square root will be essential in the proof of Naimark's dilation theorem. Given a POVM element $E_i \geq 0$, we will use its square root $\sqrt{E_i}$ to build an isometry into a larger Hilbert space.

Example: square root of a diagonal positive operator

Let

$$A = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}.$$

Then

$$A^{1/2} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix},$$

because

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^2 = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}.$$

If A is positive but not diagonal in the current basis, we diagonalize it using an orthonormal eigenbasis, take square roots of its nonnegative eigenvalues, and transform back.

3.10 The order relation $A \leq B$

For self-adjoint operators A and B on the same Hilbert space, we write

$$A \leq B$$

if

$$B - A \geq 0.$$

This is called the Loewner order or semidefinite order.

For example,

$$0 \leq A \leq I$$

means both

$$A \geq 0$$

and

$$I - A \geq 0.$$

In quantum measurement theory, an operator E satisfying

$$0 \leq E \leq I$$

is often called an effect. Effects are the individual outcome operators of POVMs. If a system is in a unit vector state ψ , then the probability associated with effect E will be

$$\langle \psi, E\psi \rangle.$$

The inequalities $0 \leq E \leq I$ guarantee that this number lies between 0 and 1.

Example: a qubit effect

Let

$$E = \begin{pmatrix} 0.8 & 0 \\ 0 & 0.2 \end{pmatrix}.$$

Then $E \geq 0$, because its eigenvalues are 0.8 and 0.2. Also,

$$I - E = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.8 \end{pmatrix} \geq 0.$$

Therefore

$$0 \leq E \leq I.$$

For a qubit vector

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle,$$

the quantity

$$\langle\psi, E\psi\rangle = 0.8|\alpha|^2 + 0.2|\beta|^2$$

is always between 0 and 1.

This is the beginning of the POVM idea: probabilities can come from positive operators that are not necessarily projections.

3.11 Isometries

An isometry is a linear map that preserves inner products.

Let

$$V : \mathcal{H} \rightarrow \mathcal{K}$$

be a linear operator between Hilbert spaces. We say that V is an isometry if

$$\langle V\phi, V\psi\rangle_{\mathcal{K}} = \langle\phi, \psi\rangle_{\mathcal{H}}$$

for all $\phi, \psi \in \mathcal{H}$.

In particular, an isometry preserves lengths:

$$\|V\psi\| = \|\psi\|.$$

In finite dimensions, V is an isometry if and only if

$$V^*V = I_{\mathcal{H}}.$$

Here $I_{\mathcal{H}}$ is the identity operator on \mathcal{H} .

To see why, observe that

$$\langle V\phi, V\psi \rangle = \langle \phi, V^*V\psi \rangle.$$

Thus V preserves all inner products precisely when

$$\langle \phi, V^*V\psi \rangle = \langle \phi, \psi \rangle$$

for all ϕ, ψ , which forces

$$V^*V = I_{\mathcal{H}}.$$

An isometry may map a smaller Hilbert space into a larger one. It preserves the geometry of the original space while embedding it into a bigger space.

Example: embedding a qubit into a qutrit

Let $\mathcal{H} = \mathbb{C}^2$ and $\mathcal{K} = \mathbb{C}^3$. Define

$$V : \mathbb{C}^2 \rightarrow \mathbb{C}^3$$

by

$$V \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}.$$

In matrix form,

$$V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then

$$V^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

and

$$V^*V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_{\mathbb{C}^2}.$$

So V is an isometry.

However,

$$VV^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which is not the identity on \mathbb{C}^3 . Instead, VV^* is the projection onto the subspace

$$\left\{ \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} : a, b \in \mathbb{C} \right\}.$$

This is a general fact:

> If $V: \mathbb{H} \rightarrow \mathbb{K}$ is an isometry, then VV^* is the orthogonal projection onto the range of V .

This fact will matter later. In Naimark dilation, the original Hilbert space sits inside a larger Hilbert space through an isometry. Operators on the larger space can then be compressed back to operators on the original space.

3.12 Unitaries as onto isometries

A unitary operator is an isometry that maps a Hilbert space onto itself.

More precisely, if

$$U : \mathcal{H} \rightarrow \mathcal{H},$$

then U is unitary if

$$U^*U = I$$

and

$$UU^* = I.$$

The first equation says U preserves inner products. The second says U reaches the whole space; no directions are missing.

Thus a unitary operator is a reversible, norm-preserving transformation. In quantum information, closed-system time evolution and ideal quantum gates are modeled by unitary operators (Nielsen and Chuang, 2010).

Example: the Hadamard operator

The Hadamard operator is

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

It satisfies

$$H^*H = I$$

and

$$HH^* = I.$$

Therefore H is unitary.

It acts on the computational basis by

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}},$$

$$H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

Unitary operators will reappear in Chapter 15, where we show how a POVM can be implemented by coupling a system to an ancilla through a unitary interaction.

3.13 Partial isometries

A partial isometry is a map that behaves like an isometry on part of a Hilbert space and sends the orthogonal complement of that part to zero.

This definition is slightly more subtle than the definition of an isometry, so let us build it carefully.

Let \mathcal{H} and \mathcal{K} be Hilbert spaces. A linear operator

$$W : \mathcal{H} \rightarrow \mathcal{K}$$

is called a partial isometry if there is a subspace $\mathcal{M} \subseteq \mathcal{H}$ such that:

1. W preserves norms on \mathcal{M} , and
2. $W\psi=0$ for every $\psi \in \mathcal{M}^\perp$.

The subspace \mathcal{M} is called the initial subspace of W . The image

$$W(\mathcal{M}) \subseteq \mathcal{K}$$

is called the final subspace of W .

Equivalently, in finite dimensions, W is a partial isometry exactly when

$$W^*W$$

is an orthogonal projection. In that case:

- W^*W is the projection onto the initial subspace,
- WW^* is the projection onto the final subspace.

Partial isometries appear naturally in operator theory and in quantum information, especially in polar decompositions and in the analysis of measurement operations (Watrous, 2018).

Example: a partial isometry on \mathbb{C}^2

Define

$$W = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then

$$W|0\rangle = |0\rangle, \quad W|1\rangle = 0.$$

So W is an isometry on the one-dimensional subspace spanned by $|0\rangle$, and it kills the orthogonal subspace spanned by $|1\rangle$.

We compute

$$W^*W = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = P_0,$$

which is an orthogonal projection. Thus W is a partial isometry.

Example: moving one subspace into another

Let $\mathcal{H} = \mathbb{C}^3$, and define W by

$$We_1 = f_1, \quad We_2 = f_2, \quad We_3 = 0,$$

where e_1, e_2, e_3 and f_1, f_2, f_3 are orthonormal bases.

Then W is an isometry from

$$\text{span}\{e_1, e_2\}$$

onto

$$\text{span}\{f_1, f_2\},$$

and it sends e_3 to zero. Its initial subspace is $\text{span}\{e_1, e_2\}$, and its final subspace is $\text{span}\{f_1, f_2\}$.

3.14 Direct sums of Hilbert spaces

The direct sum is a way to build a larger Hilbert space by placing Hilbert spaces side by side.

If \mathcal{H} and \mathcal{K} are Hilbert spaces, their direct sum is written

$$\mathcal{H} \oplus \mathcal{K}.$$

Its elements are ordered pairs

$$(\psi, \phi),$$

where $\psi \in \mathcal{H}$ and $\phi \in \mathcal{K}$.

Addition and scalar multiplication are defined componentwise:

$$(\psi_1, \phi_1) + (\psi_2, \phi_2) = (\psi_1 + \psi_2, \phi_1 + \phi_2),$$

$$a(\psi, \phi) = (a\psi, a\phi).$$

The inner product is

$$\langle (\psi_1, \phi_1), (\psi_2, \phi_2) \rangle = \langle \psi_1, \psi_2 \rangle_{\mathcal{H}} + \langle \phi_1, \phi_2 \rangle_{\mathcal{K}}.$$

Thus the norm is

$$\|(\psi, \phi)\|^2 = \|\psi\|^2 + \|\phi\|^2.$$

Example: $\mathbb{C}^2 \oplus \mathbb{C}^3$

An element of

$$\mathbb{C}^2 \oplus \mathbb{C}^3$$

has the form

$$\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \\ e \end{pmatrix} \right).$$

This space has dimension

$$2 + 3 = 5.$$

It is naturally isomorphic to \mathbb{C}^5 , but writing it as $\mathbb{C}^2 \oplus \mathbb{C}^3$ reminds us that it is built out of two pieces.

For finitely many Hilbert spaces $\mathcal{H}_1, \dots, \mathcal{H}_m$, their direct sum is

$$\bigoplus_{i=1}^m \mathcal{H}_i.$$

An element is a list

$$(\psi_1, \dots, \psi_m),$$

where $\psi_i \in \mathcal{H}_i$, and

$$\|(\psi_1, \dots, \psi_m)\|^2 = \sum_{i=1}^m \|\psi_i\|^2.$$

Direct sums are extremely important for

Document information

Chapter 3: Hilbert Spaces and Operators

Project	Naimark Dilation from First Principles
Document	Document 1.7
Author	mujirin
Verifier	Not verified
Downloaded	July 04, 2026 20:18 KST
Status	Working
Document link	https://www.theorytrace.com/projects/naimark-dilation-from-first-principles/documents/-chapter-3-hilbert-spaces-and-operators/