Naimark Dilation Theorem

Formal statement

Let mathcal H$_{(A)}$ be a finite-dimensional complex Hilbert space, and let

$$\{E_x\}_{x\in\mathcal{X}}$$

be a POVM on mathcal H$_{(A)}$. This means that each effect E_x is positive semidefinite,

$$E_x \geq 0,$$

and the effects sum to the identity operator,

$$\sum_{x\in\mathcal{X}} E_x = I_A.$$

The Naimark dilation theorem says that there exists a larger Hilbert space, an isometry

$$V : \mathcal{H}_A \longrightarrow \mathcal{H}_A \otimes \mathcal{H}_R,$$

and a projective measurement \Π$_x$\$_{(}$x∈mathcal X$_{)}$ on the larger Hilbert space such that

$$E_x = V^\dagger \Pi_x V$$

for every outcome x. Equivalently, if the original system is first embedded into the larger space by V, then a projective measurement on the enlarged system reproduces exactly the same outcome probabilities as the original POVM.

A very concrete finite-dimensional version is obtained by taking mathcal H$_{(R)}$congmathbb C$^{(}$|mathcal X|$^{)}$, with orthonormal basis \|x⟩$_{(R)}$:x∈mathcal X\, defining

$$V|\psi\rangle_A = \sum_{x\in\mathcal{X}} \sqrt{E_x}|\psi\rangle_A \otimes |x\rangle_R,$$

and taking

$$\Pi_x = I_A \otimes |x\rangle\langle x|_R.$$

Then

$$V^\dagger \Pi_x V = E_x.$$

This is the finite-dimensional form most commonly used in quantum information theory.

Proof

The proof is short, but it is conceptually important. A POVM effect E_x is positive semidefinite, so it has a positive square root √(E_x). The square-root operators are the pieces from which the dilation is built.

Introduce a reference, ancilla, or pointer system R with orthonormal basis

$$\{|x\rangle_R : x \in \mathcal{X}\}.$$

Define a linear map

$$V : \mathcal{H}_A \to \mathcal{H}_A \otimes \mathcal{H}_R$$

by

$$V|\psi\rangle_A = \sum_x \sqrt{E_x}|\psi\rangle_A \otimes |x\rangle_R.$$

We first check that V is an isometry. For arbitrary |ψ⟩,|φ⟩∈mathcal H(A),

$$
\begin{aligned}
\langle V\varphi|V\psi\rangle &= \sum_{x,y} \langle\varphi|\sqrt{E_y}\sqrt{E_x}|\psi\rangle\langle y|x\rangle_R \\
&= \sum_x \langle\varphi|E_x|\psi\rangle \\
&= \langle\varphi|\left(\sum_x E_x\right)|\psi\rangle \\
&= \langle\varphi|\psi\rangle.
\end{aligned}
$$

Thus

$$V^\dagger V = I_A,$$

so V preserves inner products and is therefore an isometry.

Now define projectors on the enlarged Hilbert space by

$$\Pi_x = I_A \otimes |x\rangle\langle x|_R.$$

They are genuine projectors because

$$\Pi_x^2 = \Pi_x, \qquad \Pi_x^\dagger = \Pi_x,$$

and they are mutually orthogonal because

$$\Pi_x \Pi_y = 0$$

when x≠ y. They also sum to the identity on the enlarged space:

$$\sum_x \Pi_x = I_A \otimes \sum_x |x\rangle\langle x|_R = I_A \otimes I_R.$$

So \Π$_x$\ is a projective measurement on mathcal H$_{(A)}$⊗mathcal H$_{(R)}$.

It remains only to show that this projective measurement reproduces the original POVM. For any |ψ⟩,

$$\Pi_x V|\psi\rangle = \sqrt{E_x}|\psi\rangle \otimes |x\rangle_R.$$

Therefore

$$\begin{aligned}\langle\psi|V^\dagger\Pi_x V|\psi\rangle &= \langle V\psi|\Pi_x|V\psi\rangle \\ &= \langle\psi|E_x|\psi\rangle.\end{aligned}$$

Since this equality holds for every vector |ψ⟩, we have the operator identity

$$V^\dagger\Pi_x V = E_x.$$

For a general mixed input state ρ₍A₎, the probability of outcome x after the enlarged projective measurement is

$$\mathrm{Tr}(\Pi_x V\rho_A V^\dagger) = \mathrm{Tr}(V^\dagger\Pi_x V\rho_A) = \mathrm{Tr}(E_x\rho_A).$$

This is exactly the Born rule for the original POVM. Hence every POVM can be realized as a projective measurement on a larger Hilbert space.

The unitary-and-ancilla implementation

The isometry version is mathematically clean. The laboratory version is often stated slightly differently. Instead of saying that V embeds mathcal H₍A₎ into mathcal H₍A₎⊗mathcal H₍R₎, we say that the system is supplied with an ancilla initialized in a fixed state |0⟩₍R₎, then a unitary is applied, and finally a projective measurement is performed.

Because V is an isometry, it can be extended to a unitary U on the larger Hilbert space, chosen so that

$$U(|\psi\rangle_A \otimes |0\rangle_R) = V|\psi\rangle_A$$

for every |ψ⟩₍A₎. After this unitary, measure the reference system in the orthonormal basis \|x⟩₍R₎\. The probability of outcome x is

$$\mathrm{Tr}\left[(I_A \otimes |x\rangle\langle x|_R)U(\rho_A \otimes |0\rangle\langle 0|_R)U^\dagger\right] = \mathrm{Tr}(E_x\rho_A).$$

This is the operational form of Naimark dilation. A generalized measurement can always be implemented by adding an ancilla, coupling the system and ancilla unitarily, and then performing an ordinary projective measurement on the larger system.

Operational meaning

The theorem says that POVMs are not mysterious new measurements that violate the projective-measurement picture. Rather, a POVM is what a projective measurement looks like when part of the measuring apparatus is included in the Hilbert space and then ignored.

The mental image is this. A projective measurement asks which orthogonal subspace a state belongs to. A POVM may have effects that are not projectors and may even have more outcomes than the dimension of the original system would allow for a projective measurement. Naimark's theorem resolves this by saying: the orthogonal subspaces exist, but not necessarily inside the original system alone. They exist in a larger system consisting of the original system plus a pointer or ancilla.

Thus a POVM is a shadow of a projective measurement. On the enlarged system, the measurement outcomes correspond to orthogonal projectors. When compressed back down to the original system, those projectors become positive operators E_x that need not be idempotent. The compression formula

$$E_x = V^\dagger \Pi_x V$$

is the entire idea in one line.

This is why the theorem is so useful in quantum information. It tells us that we are free to use POVMs as the most general measurement statistics, while still knowing that they can be physically modeled by ordinary unitary evolution and projective readout on a larger Hilbert space.

Example 1: projective measurements are already POVMs

Consider the computational-basis measurement on one qubit:

$$E_0 = |0\rangle\langle 0|, \qquad E_1 = |1\rangle\langle 1|.$$

This is both a POVM and a projective measurement. The effects are already projectors. The Naimark isometry becomes

$$V|\psi\rangle = E_0|\psi\rangle \otimes |0\rangle_R + E_1|\psi\rangle \otimes |1\rangle_R.$$

If

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle,$$

then

$$V|\psi\rangle = \alpha|0\rangle|0\rangle_R + \beta|1\rangle|1\rangle_R.$$

Measuring the reference in the $|0\rangle_{(R)}, |1\rangle_{(R)}$ basis gives outcome 0 with probability $|\alpha|^2$ and outcome 1 with probability $|\beta|^2$. This example shows that projective measurements are the special case where the POVM effects already correspond to orthogonal subspaces of the original system.

Example 2: an unsharp qubit measurement

A simple non-projective POVM is an unsharp version of the Pauli Z measurement. Fix a sharpness parameter $0 \leq \eta \leq 1$, and define

$$E_+ = \frac{1}{2}(I + \eta Z), \qquad E_- = \frac{1}{2}(I - \eta Z).$$

In the computational basis,

$$E_+ = \begin{pmatrix} \frac{1+\eta}{2} & 0 \\ 0 & \frac{1-\eta}{2} \end{pmatrix}, \qquad E_- = \begin{pmatrix} \frac{1-\eta}{2} & 0 \\ 0 & \frac{1+\eta}{2} \end{pmatrix}.$$

These operators are positive and sum to I, so they form a POVM. When $\eta=1$, this becomes the ordinary projective Z-measurement. When $\eta=0$, both effects are I/2, so the measurement output is just a fair coin toss independent of the state.

For $0<\eta<1$, the measurement is genuinely unsharp. The effects are not projectors because, for example,

$$E_+^2 \neq E_+.$$

Naimark's theorem says that this unsharp measurement can still be realized as a sharp projective measurement after introducing an ancilla. The isometry is

$$V|\psi\rangle = \sqrt{E_+}|\psi\rangle|+\rangle_R + \sqrt{E_-}|\psi\rangle|-\rangle_R.$$

Measuring the reference in the |+⟩(R),|-⟩(R) pointer basis gives the unsharp outcomes. Operationally, the ancilla has absorbed the imperfection or softness of the measurement. The final readout is projective, but because the system was first coupled to an ancilla, the induced measurement on the original qubit is not projective.

Example 3: a three-outcome qubit POVM

A qubit projective measurement cannot have three nonzero rank-one orthogonal outcomes, because a two-dimensional Hilbert space cannot contain three mutually orthogonal nonzero one-dimensional subspaces. But a qubit POVM can have three outcomes.

Consider the trine states

$$|\phi_0\rangle = |0\rangle,$$

$$|\phi_1\rangle = -\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle,$$

and

$$|\phi_2\rangle = -\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle.$$

They are three unit vectors separated symmetrically in the real X-Z plane of the Bloch sphere. Define

$$E_k = \frac{2}{3}|\phi_k\rangle\langle\phi_k|, \qquad k = 0, 1, 2.$$

One checks that

$$E_0 + E_1 + E_2 = I.$$

Thus $E_0,E_1,E_2\$ is a valid three-outcome POVM on a qubit. It is not a projective measurement on the original qubit, because the three effects are not mutually orthogonal projectors. Naimark's theorem says that the missing orthogonality appears after dilation. We introduce a three-dimensional pointer system with basis

$$|0\rangle_R, |1\rangle_R, |2\rangle_R,$$

and define

$$V|\psi\rangle = \sum_{k=0}^{2} \sqrt{E_k}|\psi\rangle|k\rangle_R.$$

Then the projective measurement $I\otimes |k\rangle\langle k|\$ on the enlarged system reproduces the trine POVM statistics.

This example shows one of the main practical reasons POVMs matter. POVMs can have outcome structures that are impossible for projective measurements on the original Hilbert space. They are especially important in state discrimination, quantum communication, and tomography, where the optimal measurement is often not a projective measurement on the system alone.

Example 4: the SIC-like tetrahedral qubit POVM

Another important example is a four-outcome qubit POVM whose effects point toward the vertices of a tetrahedron on the Bloch sphere. Let mathbf $n_1,\ldots,$mathbf n_4 be unit vectors satisfying

$$\mathbf{n}_i \cdot \mathbf{n}_j = -\frac{1}{3}$$

for $i\neq j$, and define

$$E_i = \frac{1}{4}(I + \mathbf{n}_i \cdot \boldsymbol{\sigma}), \qquad i = 1, 2, 3, 4,$$

where

$$\boldsymbol{\sigma} = (X, Y, Z)$$

is the vector of Pauli matrices. Each E_i is positive semidefinite, and the four effects sum to I. This POVM is informationally complete: from the four outcome probabilities, one can reconstruct an arbitrary qubit state.

A four-outcome informationally complete measurement cannot be a rank-one projective measurement on a single qubit, because a qubit projective measurement has at most two nonzero rank-one outcomes. Naimark dilation says that such a measurement can nevertheless be implemented by coupling the qubit to a larger pointer and making a projective measurement there.

The operational lesson is that the apparatus can carry more distinguishable pointer outcomes than the original system has orthogonal states. The POVM describes the induced statistics on the original system, while the enlarged projective measurement describes the system plus measuring device.

How to use the theorem in calculations

When given a POVM E_x\, the fastest way to build a Naimark dilation is to take square roots:

$$\sqrt{E_x}.$$

Then define

$$V = \sum_x \sqrt{E_x} \otimes |x\rangle_R.$$

This expression should be read as a map from mathcal $H_{(A)}$ to mathcal $H_{(A)}$⊗mathcal $H_{(R)}$. The condition $\sum_x E_x = I$ is exactly what guarantees that V is an isometry.

After that, the projectors are simply pointer projectors:

$$\Pi_x = I_A \otimes |x\rangle\langle x|_R.$$

The POVM is recovered by compression:

$$E_x = V^\dagger \Pi_x V.$$

If a physical circuit is desired, extend V to a unitary U acting on system plus ancilla. This is always possible in finite dimensions because an isometry maps an orthonormal basis of the input space to an orthonormal set in the output space, and that set can be completed to a full orthonormal basis. Then prepare the ancilla in $|0\rangle_{(R)}$, apply U, and projectively measure the pointer basis.

Relation to measurement operators and instruments

A POVM specifies outcome probabilities. It does not by itself uniquely specify the post-measurement state. If one chooses measurement operators M_x\ satisfying

$$E_x = M_x^\dagger M_x,$$

then one possible measurement instrument maps an input state ρ to the unnormalized conditional state

$$M_x \rho M_x^\dagger.$$

The square-root choice

$$M_x = \sqrt{E_x}$$

is only one possible choice. Another set of operators with the same M_x^† M_x=E_x gives the same outcome probabilities but may give different post-measurement states.

This distinction is important. Naimark's theorem, in its simplest POVM form, proves that the statistics of a POVM can be realized by a projective measurement on a larger space. To describe the state update as well, one must specify the full measurement instrument, not only the POVM effects.

Common mistakes

A common mistake is to think that Naimark's theorem says a POVM is already a projective measurement on the original Hilbert space. That is false. A POVM effect need not be idempotent, and different POVM effects need not be orthogonal. The theorem says that the POVM becomes projective only after moving to a larger Hilbert space.

Another common mistake is to think that the dilation is unique. It is not. Different ancilla spaces, different isometric embeddings, and different unitary extensions can produce the same POVM statistics. The theorem guarantees existence, not uniqueness.

A third mistake is to confuse the number of POVM outcomes with the dimension of the original system. A qubit POVM may have three, four, or many outcomes. The extra outcomes are made possible by the enlarged pointer system in the dilation.

A fourth mistake is to forget that the basic theorem is about measurement probabilities. The POVM E_x\ tells us

$$p(x) = \mathrm{Tr}(E_x \rho).$$

It does not uniquely tell us what the post-measurement state is. For that, one needs a choice of measurement operators or a full instrument.

Final mental image

A projective measurement is a sharp measurement of orthogonal alternatives. A POVM is what such a sharp measurement looks like when the apparatus, ancilla, or environment is partially hidden.

Naimark's theorem makes this precise:

$$E_x = V^\dagger \Pi_x V.$$

The large-space projector Π_x is the real sharp event. The isometry V embeds the original state into a larger system. The POVM effect E_x is the compressed shadow of Π_x back on the original Hilbert space.

Thus generalized measurements are operationally ordinary measurements performed after adding a quantum workspace. This is why POVMs are accepted as the general measurement formalism in quantum information theory: they are exactly the measurement statistics obtainable from projective measurements on larger systems.

References

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