Lindblad-Gorini-Kossakowski-Sudarshan Theorem

Formal statement

Let mathcal H be a finite-dimensional complex Hilbert space, and let L(mathcal H) denote the space of linear operators on mathcal H. A family of maps

$$\{\mathcal{T}_t : L(\mathcal{H}) \to L(\mathcal{H})\}_{t\geq 0}$$

is called a time-homogeneous quantum dynamical semigroup if

$$\mathcal{T}_0 = \mathrm{id}, \qquad \mathcal{T}_{t+s} = \mathcal{T}_t \mathcal{T}_s,$$

each mathcal T_t is completely positive and trace preserving, and t↦ mathcal T_t is continuous. In finite dimension, such a semigroup has a generator mathcal L satisfying

$$\mathcal{T}_t = e^{t\mathcal{L}}, \qquad \frac{d}{dt}\rho(t) = \mathcal{L}(\rho(t)).$$

The Lindblad-Gorini-Kossakowski-Sudarshan theorem says that mathcal L is the generator of such a completely positive trace-preserving semigroup if and only if it can be written in the form

$$\mathcal{L}(\rho) = -i[H,\rho] + \sum_\alpha \left(L_\alpha \rho L_\alpha^\dagger - \frac{1}{2}\{L_\alpha^\dagger L_\alpha, \rho\} \right)$$

where H=H^† is a Hermitian Hamiltonian, the operators Lα are arbitrary system operators, and

$$\{A,B\} = AB + BA$$

denotes the anticommutator. Equivalently, one often writes rates explicitly:

$$\mathcal{L}(\rho) = -i[H,\rho] + \sum_{\alpha} \gamma_{\alpha} \left(A_{\alpha}\rho A_{\alpha}^{\dagger} - \frac{1}{2}\{A_{\alpha}^{\dagger}A_{\alpha},\rho\} \right), \qquad \gamma_{\alpha} \geq 0.$$

The two forms are the same, because the rate γα can be absorbed into Lα=√(γα)Aα.

This equation is called the Lindblad master equation or the GKSL master equation. The theorem was proved independently in closely related forms by Gorini, Kossakowski, and Sudarshan for finite N-level systems, and by Lindblad in the more general language of quantum dynamical semigroups.

The meaning of “Markovian” in this theorem

The word “Markovian” is used in several ways in open quantum systems, so the theorem must be read with care. In the clean theorem above, “Markovian” means time-homogeneous semigroup evolution: the future map depends only on the elapsed time, and the maps compose as

$$\mathcal{T}_{t+s} = \mathcal{T}_t\mathcal{T}_s.$$

This is stronger and more specific than merely saying that a master equation has no explicit memory kernel. The semigroup property says that if the system evolves for time s, and then for time t, the result is exactly the same as evolving once for time t+s. There is no dependence on the detailed history before the present state.

In physics, this is the mathematical idealization behind memoryless dissipation. The environment is assumed to forget its correlations with the system quickly enough that, at the level of the reduced system, the same generator mathcal L applies at every time.

Why the Lindblad form has exactly this shape

The generator has two conceptually different parts. The first part,

$$-i[H,\rho],$$

is the familiar closed-system Schrödinger-von Neumann evolution. If the environment were absent, the density operator would satisfy

$$\frac{d\rho}{dt} = -i[H, \rho].$$

This term preserves purity and corresponds to unitary motion.

The second part,

$$\sum_\alpha \left(L_\alpha \rho L_\alpha^\dagger - \frac{1}{2}\{L_\alpha^\dagger L_\alpha, \rho\} \right),$$

is the dissipative part. The term

$$L_\alpha \rho L_\alpha^\dagger$$

looks like a Kraus branch or quantum jump. The anticommutator term is the compensating term that keeps the total trace equal to one and makes the infinitesimal evolution consistent with complete positivity.

The operational mental image is this. During a tiny time interval dt, either no visible jump happens, or one of the jumps Lα happens with probability of order dt. The no-jump branch is not simply the identity; it contains a small nonunitary correction that reduces the norm by exactly the amount needed to account for the possible jumps. When all branches are added together, trace is preserved and the resulting infinitesimal map is completely positive.

Proof that Lindblad form preserves trace

Assume

$$\mathcal{L}(\rho) = -i[H, \rho] + \sum_\alpha \left(L_\alpha \rho L_\alpha^\dagger - \frac{1}{2}\{L_\alpha^\dagger L_\alpha, \rho\} \right).$$

We first verify trace preservation at the generator level. The Hamiltonian term has zero trace because

$$\mathrm{Tr}([H, \rho]) = \mathrm{Tr}(H\rho) - \mathrm{Tr}(\rho H) = 0.$$

For one dissipative term,

$$\mathrm{Tr}(L_\alpha \rho L_\alpha^\dagger) = \mathrm{Tr}(L_\alpha^\dagger L_\alpha \rho),$$

while

$$\mathrm{Tr}\left(\frac{1}{2}\{L_\alpha^\dagger L_\alpha, \rho\}\right) = \frac{1}{2}\,\mathrm{Tr}(L_\alpha^\dagger L_\alpha \rho) + \frac{1}{2}\,\mathrm{Tr}(\rho L_\alpha^\dagger L_\alpha).$$

By cyclicity of the trace,

$$\mathrm{Tr}(\rho L_\alpha^\dagger L_\alpha) = \mathrm{Tr}(L_\alpha^\dagger L_\alpha \rho).$$

Therefore

$$\mathrm{Tr}\left[L_\alpha \rho L_\alpha^\dagger - \frac{1}{2}\{L_\alpha^\dagger L_\alpha, \rho\}\right] = 0.$$

Thus

$$\mathrm{Tr}\,\mathcal{L}(\rho) = 0.$$

Consequently, if ρ(t) solves

$$\frac{d\rho}{dt} = \mathcal{L}(\rho),$$

then

$$\frac{d}{dt}\,\mathrm{Tr}\,\rho(t) = 0.$$

The Lindblad generator preserves normalization.

Why the Lindblad form gives completely positive evolution

Trace preservation alone is not enough. A physically valid quantum evolution must remain positive even when the system is entangled with an arbitrary reference. This is why complete positivity is essential.

To see why Lindblad form gives complete positivity, consider a very small time step dt. Define approximate Kraus operators

$$K_0 = I - \left(iH + \frac{1}{2} \sum_\alpha L_\alpha^\dagger L_\alpha \right) dt$$

and

$$K_\alpha = \sqrt{dt}\, L_\alpha.$$

Now form the infinitesimal channel

$$\mathcal{T}_{dt}(\rho) = K_0 \rho K_0^\dagger + \sum_\alpha K_\alpha \rho K_\alpha^\dagger.$$

Expanding to first order in dt, we obtain

$$K_0 \rho K_0^\dagger = \rho - i[H, \rho] dt - \frac{1}{2} \sum_\alpha \{L_\alpha^\dagger L_\alpha, \rho\} dt + O(dt^2),$$

and

$$\sum_\alpha K_\alpha \rho K_\alpha^\dagger = \sum_\alpha L_\alpha \rho L_\alpha^\dagger dt.$$

Thus

$$\mathcal{T}_{dt}(\rho) = \rho + dt\, \mathcal{L}(\rho) + O(dt^2).$$

The map mathcal T₍dt₎ is completely positive because it has Kraus form. It is trace preserving to first order, and the exact semigroup e⁽tmathcal L⁾ is obtained as the limit of many small steps:

$$e^{t\mathcal{L}} = \lim_{n\to\infty}\left(I + \frac{t}{n}\mathcal{L}\right)^n,$$

with the rigorous version obtained by replacing the first-order approximation by completely positive approximants and taking the finite-dimensional norm limit. Since complete positivity is closed under composition and norm limits in finite dimension, $e^{(\text{tmathcal L})}$ is completely positive. Since the trace derivative vanishes, it is trace preserving.

This proves the easier direction: every generator in Lindblad form produces a completely positive trace-preserving semigroup.

Why every Markovian generator must have Lindblad form

The converse is the deeper part of the theorem. Suppose

$$\mathcal{T}_t = e^{t\mathcal{L}}$$

is a completely positive trace-preserving semigroup. For each t≥0, mathcal T_t is a quantum channel. Hence it has a Choi matrix

$$J(\mathcal{T}_t) \geq 0.$$

At t=0, the map is the identity channel, so

$$J(\mathcal{T}_0) = |\Omega\rangle\langle\Omega|.$$

The generator mathcal L is the infinitesimal change of the channel at t=0:

$$\mathcal{T}_t = I + t\mathcal{L} + o(t).$$

Therefore

$$J(\mathcal{T}_t) = J(\mathrm{id}) + tJ(\mathcal{L}) + o(t).$$

The positivity of J(mathcal T_t) for all small t>0 imposes strong constraints on J(mathcal L). On the subspace orthogonal to the maximally entangled vector |Ω⟩, the identity-channel Choi matrix has zero support. Therefore the first-order correction must be positive on that orthogonal subspace. This property is often called conditional complete positivity.

When this infinitesimal Choi-positivity condition is written in an operator basis, it produces a positive semidefinite coefficient matrix, called the Kossakowski matrix. Choose an orthonormal operator basis

$$F_0 = \frac{I}{\sqrt{d}}, \qquad F_1, \ldots, F_{d^2-1},$$

where F_1,...,$F_{(d^2-1)}$ are traceless. The dissipative part of the generator can be written as

$$\sum_{\mu,\nu=1}^{d^2-1} c_{\mu\nu} \left(F_\mu \rho F_\nu^\dagger - \frac{1}{2}\{F_\nu^\dagger F_\mu, \rho\} \right),$$

where the matrix

$$C = (c_{\mu\nu})$$

is positive semidefinite. The remaining skew-Hermitian freedom becomes a Hamiltonian commutator

$$-i[H, \rho].$$

Since C≥0, it can be diagonalized:

$$C = U^\dagger D U, \qquad D_{\alpha\alpha} = \gamma_\alpha \geq 0.$$

Defining new operators

$$L_\alpha = \sqrt{\gamma_\alpha} \sum_\mu U_{\alpha\mu} F_\mu,$$

the generator becomes

$$\mathcal{L}(\rho) = -i[H,\rho] + \sum_{\alpha} \left(L_\alpha \rho L_\alpha^\dagger - \frac{1}{2}\{L_\alpha^\dagger L_\alpha, \rho\} \right).$$

This is the Lindblad form. The converse proof therefore rests on one central fact: infinitesimal complete positivity forces the dissipative coefficient matrix to be positive semidefinite, and trace preservation forces the anticommutator correction. Once those two constraints are imposed, nothing else is possible.

Equivalent Kossakowski-matrix form

The diagonal Lindblad form is not the only useful expression. In a fixed traceless operator basis Fμ\, the generator may be written as

$$\mathcal{L}(\rho) = -i[H,\rho] + \sum_{\mu,\nu} c_{\mu\nu} \left(F_\mu \rho F_\nu^\dagger - \frac{1}{2}\{F_\nu^\dagger F_\mu, \rho\} \right),$$

with

$$C = (c_{\mu\nu}) \geq 0.$$

This is often called the Gorini-Kossakowski-Sudarshan form. Diagonalizing the positive matrix C gives the more familiar Lindblad-operator form. The Kossakowski matrix makes complete positivity visible as an ordinary matrix positivity condition.

This form is especially useful for qubits. If one chooses the Pauli basis, the entries of the Kossakowski matrix describe decoherence rates and correlations between noise directions. Positivity of C gives inequalities among the physical rates. This is exactly the kind of structure analyzed in the original N-level GKS paper.

Example 1: closed-system unitary evolution

If there is no dissipative part, then

$$\mathcal{L}(\rho) = -i[H,\rho].$$

The solution is

$$\rho(t) = e^{-iHt}\rho(0)e^{iHt}.$$

This is the ordinary unitary evolution of a closed quantum system. The Lindblad theorem includes this as the special case with no jump operators.

Operationally, this means no information leaks to the environment. The state may rotate in Hilbert space, but purity is preserved. If $\rho(0)=|\psi\rangle\langle\psi|$, then $\rho(t)$ remains a rank-one projector.

Example 2: pure dephasing of a qubit

Let

$$L = \sqrt{\gamma}\, Z, \qquad H = 0.$$

The Lindblad equation becomes

$$\frac{d\rho}{dt} = \gamma\left(Z\rho Z - \rho\right),$$

because

$$Z^\dagger Z = I.$$

Write

$$\rho = \begin{pmatrix} a & b \\ \bar{b} & 1-a \end{pmatrix}.$$

Then

$$Z\rho Z = \begin{pmatrix} a & -b \\ -\bar{b} & 1-a \end{pmatrix}.$$

Therefore

$$\frac{d\rho}{dt} = \gamma \begin{pmatrix} 0 & -2b \\ -2\bar{b} & 0 \end{pmatrix}.$$

The populations a and 1-a do not change, while the coherence satisfies

$$\frac{db}{dt} = -2\gamma b.$$

Hence

$$b(t) = e^{-2\gamma t} b(0).$$

This is pure dephasing. The environment learns phase-sensitive information about the computational-basis label, and the off-diagonal terms decay. The diagonal probabilities remain unchanged.

The operational image is simple. Dephasing does not exchange energy with the system. It destroys coherence between $|0\rangle$ and $|1\rangle$ while leaving their populations fixed.

Example 3: spontaneous emission and amplitude damping

Let

$$L = \sqrt{\gamma}\,\sigma_-, \qquad \sigma_- = |0\rangle\langle 1|, \qquad H = 0.$$

The master equation is

$$\frac{d\rho}{dt} = \gamma \left(\sigma_- \rho \sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho\} \right),$$

where

$$\sigma_+ = |1\rangle\langle 0|, \qquad \sigma_+\sigma_- = |1\rangle\langle 1|.$$

Write

$$\rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix}.$$

Then the excited-state population obeys

$$\frac{d\rho_{11}}{dt} = -\gamma\rho_{11},$$

so

$$\rho_{11}(t) = e^{-\gamma t}\rho_{11}(0).$$

The ground-state population increases as

$$\rho_{00}(t) = 1 - \rho_{11}(t),$$

and the coherence decays as

$$\rho_{01}(t) = e^{-\gamma t/2}\rho_{01}(0).$$

This is the continuous-time version of the amplitude damping channel. The jump operator σ_- describes the transition

$$|1\rangle \to |0\rangle.$$

The environment receives the emitted excitation. If we do not observe the environment, the system evolves irreversibly toward the ground state.

Example 4: depolarization of a qubit

A symmetric depolarizing Lindblad generator can be written as

$$\mathcal{L}(\rho) = \gamma \sum_{j=x,y,z} (\sigma_j \rho \sigma_j - \rho).$$

Here the jump operators are proportional to the Pauli matrices. Write the qubit state in Bloch-vector form:

$$\rho = \frac{1}{2}(I + \mathbf{r} \cdot \boldsymbol{\sigma}).$$

Using the Pauli identities, one obtains

$$\frac{d\mathbf{r}}{dt} = -4\gamma\,\mathbf{r}.$$

Thus

$$\mathbf{r}(t) = e^{-4\gamma t}\mathbf{r}(0).$$

The state moves radially toward the center of the Bloch ball:

$$\rho(t) \to \frac{I}{2}.$$

Operationally, depolarization is isotropic loss of information. Unlike pure dephasing, which destroys only certain coherences, depolarization erases all directional information in the qubit state.

Example 5: thermal relaxation with upward and downward jumps

A two-level atom coupled to a thermal bath can be modeled with two jump operators:

$$L_- = \sqrt{\gamma_-}\,\sigma_-, \qquad L_+ = \sqrt{\gamma_+}\,\sigma_+.$$

The downward jump σ_- represents emission into the bath, while the upward jump σ_+ represents absorption from the bath. The master equation is

$$\frac{d\rho}{dt} = \gamma_-\left(\sigma_-\rho\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho\}\right) + \gamma_+\left(\sigma_+\rho\sigma_- - \frac{1}{2}\{\sigma_-\sigma_+, \rho\}\right).$$

The excited-state population satisfies

$$\frac{d\rho_{11}}{dt} = -\gamma_- \rho_{11} + \gamma_+ \rho_{00}.$$

Since

$$\rho_{00} = 1 - \rho_{11},$$

we get

$$\frac{d\rho_{11}}{dt} = -(\gamma_- + \gamma_+)\rho_{11} + \gamma_+.$$

The stationary excited-state population is

$$\rho_{11}^{(\infty)} = \frac{\gamma_+}{\gamma_- + \gamma_+}.$$

This example shows how Lindblad generators describe relaxation toward a steady state. The steady state is determined by the balance between upward and downward transition rates.

Quantum trajectories and the jump picture

The Lindblad equation can be read in two complementary ways. As a deterministic master equation, it evolves the density operator smoothly. But it can also be unraveled into random pure-state trajectories. In such a trajectory picture, the system evolves under an effective non-Hermitian Hamiltonian between jumps,

$$H_{\text{eff}} = H - \frac{i}{2} \sum_\alpha L_\alpha^\dagger L_\alpha,$$

and occasionally undergoes a jump

$$|\psi\rangle \mapsto \frac{L_\alpha |\psi\rangle}{\| L_\alpha |\psi\rangle \|}.$$

Averaging many such random trajectories reproduces the density-matrix Lindblad equation.

This is not an additional assumption in the theorem. It is an interpretation of the same equation. It gives a useful operational picture of continuous monitoring: if the environment is monitored, one may see individual jumps; if the environment is ignored, one sees the averaged Lindblad evolution.

Non-uniqueness of Lindblad operators

The Lindblad operators are not unique. If we rotate them by a unitary matrix,

$$\widetilde{L}_\mu = \sum_\alpha u_{\mu\alpha} L_\alpha,$$

then the dissipator remains the same. There is also freedom to shift Lindblad operators by multiples of the identity, provided the Hamiltonian is adjusted accordingly. This non-uniqueness is the infinitesimal analogue of non-uniqueness of Kraus operators.

Therefore one should not think of a particular set of Lα's as uniquely determined physical events unless a measurement scheme or microscopic model has selected a particular unraveling. The generator mathcal L is physical; the chosen Lindblad representation may be partly representational.

Relation to Kraus, Stinespring, and Choi theory

The Lindblad theorem is the continuous-time analogue of the Kraus and Stinespring representation theorems.

For a finite time t, the map

$$\mathcal{T}_t = e^{t\mathcal{L}}$$

is a quantum channel. Hence it has Kraus operators, a Stinespring dilation, and a Choi matrix. The Lindblad theorem asks a sharper question: which linear maps mathcal L can appear as the infinitesimal generator of such channels for every t≥0?

The answer is not “any trace-preserving Hermiticity-preserving linear map.” The generator must have the GKSL structure. The dissipative coefficient matrix must be positive semidefinite. That positivity is the infinitesimal remnant of complete positivity of mathcal T_t.

Thus the hierarchy is:

$$\text{Kraus/Stinespring/Choi} \quad \text{describe one channel,}$$

while

$$\text{GKSL} \quad \text{describes a continuous semigroup of channels.}$$

Common mistakes

A common mistake is to say that every Markovian-looking master equation is physically valid. It is not. A master equation may preserve trace and Hermiticity but still fail to preserve positivity or complete positivity. The Lindblad form is the condition that prevents this failure for time-homogeneous semigroup dynamics.

A second mistake is to forget the word semigroup. The theorem does not say that every open-system evolution has a time-independent Lindblad generator. Many physical environments have memory, strong coupling, structured spectra, or initial system-environment correlations. Such cases may lead to non-Markovian dynamics that cannot be represented by a fixed GKSL generator.

A third mistake is to assume that time-dependent equations of the form

$$\frac{d\rho}{dt} = -i[H(t),\rho] + \sum_\alpha \gamma_\alpha(t) \left(A_\alpha(t)\rho A_\alpha^\dagger(t) - \frac{1}{2}\{A_\alpha^\dagger(t)A_\alpha(t),\rho\} \right)$$

are exactly the original semigroup theorem. They are related, but they describe a different setting. If all instantaneous rates are nonnegative, the dynamics is often CP-divisible and is called time-local Markovian in a broader sense. But the original GKSL theorem concerns time-homogeneous semigroups with a time-independent generator.

A fourth mistake is to interpret Lindblad operators as uniquely real physical jumps. They can often be given that meaning in a chosen measurement scheme, but mathematically they are not unique. Different Lindblad representations can define the same generator.

Final mental image

The Lindblad-Gorini-Kossakowski-Sudarshan theorem says that memoryless continuous quantum evolution has only one possible structure. It must be generated by a Hamiltonian part plus dissipative jump terms:

$$\mathcal{L}(\rho) = -i[H, \rho] + \sum_{\alpha} \left(L_\alpha \rho L_\alpha^\dagger - \frac{1}{2}\{L_\alpha^\dagger L_\alpha, \rho\} \right).$$

The Hamiltonian term rotates the state coherently. The jump terms describe ways information, energy, or coherence can leak into an environment. The anticommutator terms keep the probabilities normalized and make the infinitesimal process compatible with complete positivity.

Operationally, the theorem says that if an open quantum system evolves continuously, memorylessly, and physically, then its generator must look like a balance between coherent motion, possible jumps, and no-jump normalization. This is why the GKSL equation is the central master equation of Markovian open quantum systems.

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