Choi-Jamiołkowski Isomorphism

Formal statement

Let mathcal H(A) and mathcal H(B) be finite-dimensional complex Hilbert spaces, with

$$d_A = \dim \mathcal{H}_A.$$

Let

$$\mathcal{N} : L(\mathcal{H}_A) \to L(\mathcal{H}_B)$$

be a linear map. Choose an orthonormal basis \|i⟩(A)\i=1(d(A)) for the input space and introduce a copy A' of the input system. Define the unnormalized maximally entangled vector

$$|\Omega\rangle_{A'A} = \sum_{i=1}^{d_A} |i\rangle_{A'} |i\rangle_A.$$

The Choi operator, or Choi matrix, of mathcal N is

$$J(\mathcal{N}) = (I_{A'} \otimes \mathcal{N})(|\Omega\rangle\langle\Omega|_{A'A}).$$

Equivalently,

$$J(\mathcal{N}) = \sum_{i,j=1}^{d_A} |i\rangle\langle j|_{A'} \otimes \mathcal{N}(|i\rangle\langle j|)_B.$$

The Choi-Jamiołkowski isomorphism says that the linear map mathcal N is completely determined by this bipartite operator. The inverse formula is

$$\mathcal{N}(X) = \mathrm{Tr}_{A'} \left[(X^T \otimes I_B) J(\mathcal{N}) \right],$$

where X(T) is the transpose in the chosen input basis.

With this unnormalized convention, the channel conditions become especially simple:

$$\mathcal{N} \text{ is completely positive} \quad \Longleftrightarrow \quad J(\mathcal{N}) \geq 0,$$

and

$$\mathcal{N} \text{ is trace preserving} \quad \Longleftrightarrow \quad \operatorname{Tr}_B J(\mathcal{N}) = I_{A'}.$$

Therefore mathcal N is a quantum channel if and only if

$$J(\mathcal{N}) \geq 0, \qquad \operatorname{Tr}_B J(\mathcal{N}) = I_{A'}.$$

Some authors use the normalized maximally entangled vector

$$|\Phi\rangle_{A'A} = \frac{1}{\sqrt{d_A}} \sum_i |i\rangle_{A'} |i\rangle_A.$$

Then the normalized Choi state is

$$\omega_{\mathcal{N}} = (I_{A'} \otimes \mathcal{N})(|\Phi\rangle\langle\Phi|) = \frac{1}{d_A} J(\mathcal{N}).$$

In that convention,

$$\operatorname{Tr}_B \omega_{\mathcal{N}} = \frac{I_{A'}}{d_A}.$$

Both conventions are common. The mathematics is the same, but one must not mix the normalization factors.

Meaning before the proof

The theorem says that a quantum channel can be studied as a bipartite operator. Instead of thinking of mathcal N as a machine that accepts an input state and produces an output state, we may feed half of a maximally entangled state into the machine and record the joint state of the untouched reference and the output. That joint object is the Choi matrix.

The operational mental image is this. A maximally entangled input contains all possible input matrix elements coherently. If the channel acts on one half of this entangled state, the output remembers how the channel acts on every basis operator |i⟩⟨ j|. Thus the Choi matrix is like a complete fingerprint of the channel. Nothing is lost: from the Choi matrix one can reconstruct the action of the channel on every input operator.

This is why the Choi-Jamiołkowski isomorphism is sometimes called channel-state duality. Strictly speaking, with the unnormalized convention the Choi object has trace d(A), not trace one. With the normalized convention, it is a genuine density operator whenever mathcal N is a channel. But not every bipartite density operator is the normalized Choi state of a channel. The marginal condition

$$\mathrm{Tr}_B\, \omega_{\mathcal{N}} = \frac{I_{A'}}{d_A}$$

must also hold.

Proof that the Choi matrix determines the map

Start from the definition

$$J(\mathcal{N}) = \sum_{i,j} |i\rangle\langle j|_{A'} \otimes \mathcal{N}(|i\rangle\langle j|)_B.$$

Let

$$X = \sum_{k,\ell} X_{k\ell} |k\rangle\langle \ell|$$

be an arbitrary operator on mathcal H(A). In the chosen basis,

$$X^T = \sum_{k,\ell} X_{k\ell} |\ell\rangle\langle k|.$$

Now compute

$$\mathrm{Tr}_{A'}[(X^T \otimes I_B) J(\mathcal{N})].$$

Substituting the expansion of J(mathcal N), we get

$$\begin{aligned}(X^T \otimes I_B)J(\mathcal{N}) &= \sum_{k,\ell}\sum_{i,j} X_{k\ell}|\ell\rangle\langle k|i\rangle\langle j|_{A'} \otimes \mathcal{N}(|i\rangle\langle j|) \\ &= \sum_{k,\ell}\sum_{i,j} X_{k\ell}\delta_{ki}|\ell\rangle\langle j|_{A'} \otimes \mathcal{N}(|i\rangle\langle j|) \\ &= \sum_{k,\ell,j} X_{k\ell}|\ell\rangle\langle j|_{A'} \otimes \mathcal{N}(|k\rangle\langle j|).\end{aligned}$$

Taking the partial trace over A', we use

$$\mathrm{Tr}(|\ell\rangle\langle j|) = \delta_{\ell j}.$$

Therefore

$$\begin{aligned}\mathrm{Tr}_{A'}[(X^T \otimes I_B)J(\mathcal{N})] &= \sum_{k,\ell,j} X_{k\ell}\delta_{\ell j}\mathcal{N}(|k\rangle\langle j|) \\ &= \sum_{k,\ell} X_{k\ell}\mathcal{N}(|k\rangle\langle \ell|) \\ &= \mathcal{N}\left(\sum_{k,\ell} X_{k\ell}|k\rangle\langle \ell|\right) \\ &= \mathcal{N}(X).\end{aligned}$$

Thus the inverse formula holds:

$$\mathcal{N}(X) = \mathrm{Tr}_{A'}[(X^T \otimes I_B)J(\mathcal{N})].$$

So the Choi matrix contains exactly the same information as the original linear map.

Complete positivity as positivity of the Choi matrix

We now prove the central structural statement:

$$\mathcal{N} \text{ is completely positive} \quad \Longleftrightarrow \quad J(\mathcal{N}) \geq 0.$$

First suppose mathcal N is completely positive. The operator

$$|\Omega\rangle\langle\Omega|$$

is positive semidefinite. Complete positivity means that $I_{(A')}$⊗mathcal N maps positive semidefinite operators to positive semidefinite operators. Therefore

$$J(\mathcal{N}) = (I_{A'} \otimes \mathcal{N})(|\Omega\rangle\langle\Omega|) \geq 0.$$

The converse is more interesting. Suppose

$$J(\mathcal{N}) \geq 0.$$

By the spectral theorem, we may write

$$J(\mathcal{N}) = \sum_k |v_k\rangle\langle v_k|_{A'B},$$

where each $|v_k\rangle$ absorbs the square root of a nonzero eigenvalue. Each vector $|v_k\rangle$∈mathcal $H_{(A')}$⊗mathcal $H_{(B)}$ can be reshaped into an operator

$$E_k : \mathcal{H}_A \to \mathcal{H}_B$$

by the rule

$$|v_k\rangle = (I_{A'} \otimes E_k)|\Omega\rangle.$$

Equivalently, if

$$|v_k\rangle = \sum_i |i\rangle_{A'} \otimes |b_{k,i}\rangle_B,$$

then

$$E_k|i\rangle_A = |b_{k,i}\rangle_B.$$

Using the inverse formula, the map associated with J(mathcal N) is

$$\mathcal{N}(X) = \sum_k E_k X E_k^\dagger.$$

This is a Kraus representation, so mathcal N is completely positive. Hence positivity of the Choi matrix is exactly complete positivity of the map.

This result is often called Choi's theorem. Jamiołkowski introduced the correspondence between maps and bipartite operators in 1972, while Choi's 1975 paper gave the finite-dimensional complete-positivity criterion now used throughout quantum information theory.

Trace preservation as a partial trace condition

Now we prove the trace-preserving condition. Starting from

$$J(\mathcal{N}) = \sum_{i,j} |i\rangle\langle j|_{A'} \otimes \mathcal{N}(|i\rangle\langle j|)_B,$$

take the partial trace over B:

$$\mathrm{Tr}_B\, J(\mathcal{N}) = \sum_{i,j} |i\rangle\langle j|_{A'}\, \mathrm{Tr}[\mathcal{N}(|i\rangle\langle j|)].$$

If mathcal N is trace preserving, then

$$\mathrm{Tr}[\mathcal{N}(|i\rangle\langle j|)] = \mathrm{Tr}(|i\rangle\langle j|) = \delta_{ij}.$$

Therefore

$$\mathrm{Tr}_B\, J(\mathcal{N}) = \sum_i |i\rangle\langle i|_{A'} = I_{A'}.$$

Conversely, if

$$\mathrm{Tr}_B\, J(\mathcal{N}) = I_{A'},$$

then for every input operator X, using the inverse formula gives

$$\begin{aligned}\mathrm{Tr}[\mathcal{N}(X)] &= \mathrm{Tr}\left(\mathrm{Tr}_{A'}[(X^T \otimes I_B)J(\mathcal{N})]\right) \\ &= \mathrm{Tr}[(X^T \otimes I_B)J(\mathcal{N})] \\ &= \mathrm{Tr}\left[X^T \, \mathrm{Tr}_B \, J(\mathcal{N})\right] \\ &= \mathrm{Tr}(X^T I_{A'}) \\ &= \mathrm{Tr}(X).\end{aligned}$$

So mathcal N is trace preserving. Hence

$$\mathcal{N} \text{ is trace preserving} \quad \Longleftrightarrow \quad \mathrm{Tr}_B \, J(\mathcal{N}) = I_{A'}.$$

The same reasoning gives useful variations. A completely positive map is trace nonincreasing exactly when

$$\mathrm{Tr}_B \, J(\mathcal{N}) \leq I_{A'}.$$

A channel is unital, meaning mathcal N(I(A))=I(B), exactly when

$$\mathrm{Tr}_{A'} \, J(\mathcal{N}) = I_B.$$

Trace preservation and unitality are different conditions. The Choi matrix makes this difference visible as two different partial traces.

Relation to Kraus operators

The Choi matrix and Kraus representation are the same information in two different forms. If

$$\mathcal{N}(\rho) = \sum_k E_k \rho E_k^\dagger,$$

then

$$J(\mathcal{N}) = \sum_k |E_k\rangle\!\rangle\langle\!\langle E_k|,$$

where

$$|E_k\rangle\rangle = (I_{A'} \otimes E_k)|\Omega\rangle = \sum_i |i\rangle_{A'} E_k|i\rangle_B$$

is the vectorization of the Kraus operator. Thus a Kraus decomposition becomes a decomposition of the Choi matrix into positive rank-one terms.

Conversely, if

$$J(\mathcal{N}) = \sum_k |v_k\rangle\langle v_k|,$$

then reshaping each $|v_k\rangle$ gives a Kraus operator E_k. A minimal Kraus representation is obtained by diagonalizing J(mathcal N) and keeping only the nonzero eigenvalues. Therefore the minimal number of Kraus operators equals

$$\operatorname{rank} J(\mathcal{N}),$$

called the Choi rank of the channel.

Operationally, the Choi matrix stores the coherent sum of all possible Kraus branches. Diagonalizing it chooses one particular environment basis, and that basis choice produces one particular Kraus representation.

Example 1: identity channel

Let

$$\mathrm{id}(\rho) = \rho$$

on a d-dimensional system. Its Choi matrix is

$$J(\mathrm{id}) = (I \otimes \mathrm{id})(|\Omega\rangle\langle\Omega|) = |\Omega\rangle\langle\Omega|.$$

Thus the identity channel corresponds to a rank-one Choi matrix. The normalized Choi state is

$$|\Phi\rangle\langle\Phi|, \qquad |\Phi\rangle = \frac{1}{\sqrt{d}} \sum_i |i\rangle|i\rangle.$$

This is maximally entangled. The interpretation is beautiful: a perfect quantum channel preserves the entanglement between the input and a reference. If one half of a maximally entangled state is sent through the identity channel, the reference and output remain maximally entangled.

The Choi rank is one, corresponding to the single Kraus operator

$$E_1 = I.$$

So the identity channel is noiseless and needs no environment branch.

Example 2: completely depolarizing channel

Consider the d-dimensional completely depolarizing channel

$$\mathcal{D}(\rho) = \mathrm{Tr}(\rho)\frac{I}{d}.$$

For a matrix unit $|i\rangle\langle j|$,

$$\mathcal{D}(|i\rangle\langle j|) = \delta_{ij}\frac{I}{d}.$$

Therefore

$$\begin{aligned} J(\mathcal{D}) &= \sum_{i,j} |i\rangle\langle j| \otimes \mathcal{D}(|i\rangle\langle j|) \\ &= \sum_{i} |i\rangle\langle i| \otimes \frac{I}{d} \\ &= \frac{I \otimes I}{d}. \end{aligned}$$

The normalized Choi state is

$$\omega_{\mathcal{D}} = \frac{I \otimes I}{d^2},$$

which is maximally mixed on the reference-output system. This is the opposite of the identity-channel example. The identity channel preserves maximal entanglement with the reference. The completely depolarizing channel destroys all such correlation, leaving a maximally mixed Choi state.

This example gives a strong operational image. Looking at the Choi state, a good quantum channel is one that preserves reference-output correlation. A completely noisy channel breaks that correlation entirely.

Example 3: complete dephasing

For a qubit, the complete dephasing channel in the computational basis is

$$\Delta(\rho) = |0\rangle\langle 0|\rho|0\rangle\langle 0| + |1\rangle\langle 1|\rho|1\rangle\langle 1|.$$

It acts on matrix units as

$$\Delta(|0\rangle\langle 0|) = |0\rangle\langle 0|, \qquad \Delta(|1\rangle\langle 1|) = |1\rangle\langle 1|,$$

but

$$\Delta(|0\rangle\langle 1|) = 0, \qquad \Delta(|1\rangle\langle 0|) = 0.$$

Therefore

$$J(\Delta) = |00\rangle\langle 00| + |11\rangle\langle 11|.$$

The normalized Choi state is

$$\omega_\Delta = \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11|.$$

This state has classical correlation between reference and output, but no coherence between $|00\rangle$ and $|11\rangle$. Compare this with the identity channel, whose normalized Choi state is

$$|\Phi^+\rangle\langle\Phi^+| = \frac{1}{2}(|00\rangle + |11\rangle)(\langle 00| + \langle 11|).$$

The dephasing channel keeps the classical label but destroys the phase coherence. The Choi matrix shows this visually: the diagonal correlation remains, while the off-diagonal entangled coherence disappears.

Example 4: amplitude damping channel

The qubit amplitude damping channel with damping probability γ has Kraus operators

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \qquad E_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}.$$

Using

$$J(\mathcal{N}) = \sum_k |E_k\rangle\rangle\langle\langle E_k|,$$

with basis order

$$|00\rangle, |01\rangle, |10\rangle, |11\rangle,$$

we obtain

$$J(\mathcal{N}_\gamma) = \begin{pmatrix} 1 & 0 & 0 & \sqrt{1-\gamma} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ \sqrt{1-\gamma} & 0 & 0 & 1-\gamma \end{pmatrix}.$$

One can check trace preservation directly:

$$\mathrm{Tr}_B\, J(\mathcal{N}_\gamma) = I.$$

This Choi matrix also shows that amplitude damping is not simply classical random Pauli noise. It has a structured coherence between $|00\rangle$ and $|11\rangle$, but it also contains the decay branch $|10\rangle\langle 10|$. The Choi matrix records both the coherent no-jump part and the irreversible decay part.

Example 5: transpose map and why complete positivity matters

Define the transpose map

$$T(X) = X^T$$

on a d-dimensional system. The transpose map is positive: if $X \geq 0$, then $X^{(T)} \geq 0$. But it is not completely positive.

Its Choi matrix is

$$J(T) = \sum_{i,j} |i\rangle\langle j| \otimes |j\rangle\langle i|.$$

This is the swap operator

$$F = \sum_{i,j} |ij\rangle\langle ji|.$$

For $d \geq 2$, the swap operator has negative eigenvalues on the antisymmetric subspace. For example, for two qubits,

$$|\psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

satisfies

$$F|\psi^-\rangle = -|\psi^-\rangle.$$

Therefore

$$J(T) \ngeq 0.$$

By Choi's theorem, the transpose map is not completely positive. This is one of the cleanest examples showing why ordinary positivity is not enough for quantum channels. A physical operation may act on part of an entangled state, so it must remain positive even after tensoring with an identity map on an arbitrary reference system.

How to use the Choi isomorphism

The Choi matrix is one of the most useful computational tools in quantum information. If a channel is given abstractly, the Choi matrix gives a concrete matrix representation. To test whether a proposed linear map is physically valid, compute J(mathcal N). The map is completely positive exactly when J(mathcal N)≥0. It is trace preserving exactly when Tr(B) J(mathcal N)=I. Thus the difficult-looking condition of complete positivity becomes an ordinary semidefinite constraint.

This is why the Choi representation is central in semidefinite programming formulations of quantum information tasks. Channel discrimination, diamond-norm optimization, entanglement-assisted communication, quantum combs, process tomography, and many one-shot information problems are naturally written using Choi operators.

The Choi matrix is also used experimentally in quantum process tomography. If one reconstructs the Choi matrix of an unknown process, one has reconstructed the process itself. Positivity of the reconstructed Choi matrix enforces complete positivity, and the partial trace condition enforces trace preservation.

Basis dependence and what is invariant

The construction uses a chosen input basis because the maximally entangled vector

$$|\Omega\rangle = \sum_i |i\rangle|i\rangle$$

and the transpose in the inverse formula depend on that basis. This does not mean the physical channel depends on an arbitrary basis. It means that the matrix representation of the channel depends on a coordinate choice, just as an ordinary matrix representation of a linear operator depends on a basis.

If we change the input basis, the Choi matrix changes by a corresponding unitary conjugation and transpose convention. Properties such as positivity, Choi rank, trace preservation, and channel action are invariant. The basis dependence is therefore representational, not physical.

Common mistakes

A common mistake is to forget the normalization convention. With the unnormalized vector $|\Omega\rangle$, a channel has

$$\operatorname{Tr} J(\mathcal{N}) = d_A.$$

With the normalized vector $|\Phi\rangle$, the Choi object has trace one and is a genuine density operator. Many formula errors come from mixing these two conventions.

A second mistake is to think every bipartite state is a channel. Under the normalized convention, the Choi object of a channel is a bipartite density operator, but it must satisfy

$$\mathrm{Tr}_B\,\omega_{\mathcal{N}} = \frac{I_{A'}}{d_A}.$$

A bipartite state with the wrong marginal does not represent a trace-preserving channel.

A third mistake is to confuse positivity of the map with positivity of the Choi matrix. Positivity of the Choi matrix is equivalent to complete positivity, not merely positivity. The transpose map shows the difference: it is positive as a map on one system, but its Choi matrix has negative eigenvalues, so it is not a valid quantum channel.

A fourth mistake is to ignore the transpose in the inverse formula

$$\mathcal{N}(X) = \mathrm{Tr}_{A'}[(X^T \otimes I_B)J(\mathcal{N})].$$

The transpose is not decoration. It comes from the chosen convention for vectorizing operators. Different vectorization conventions may move the transpose elsewhere, but some convention-dependent transpose or index ordering must be handled correctly.

Final mental image

The Choi-Jamiołkowski isomorphism says that a quantum channel can be converted into a bipartite operator by sending half of a maximally entangled state through the channel:

$$J(\mathcal{N}) = (I \otimes \mathcal{N})(|\Omega\rangle\langle\Omega|).$$

The reference half remembers the input indices. The output half records what the channel does to them. Together they form a complete fingerprint of the process.

In this representation, the abstract physical requirements of a channel become simple matrix statements:

$$\text{complete positivity} \quad \Longleftrightarrow \quad J(\mathcal{N}) \geq 0,$$

and

$$\text{trace preservation} \quad \Longleftrightarrow \quad \mathrm{Tr}_B\, J(\mathcal{N}) = I.$$

Thus the theorem turns channel theory into operator theory on a bipartite Hilbert space. This is why it is one of the central technical tools of quantum information theory.

References

Jamiołkowski, Andrzej. “Linear Transformations Which Preserve Trace and Positive Semidefiniteness of Operators.” Reports on Mathematical Physics 3, no. 4 (1972): 275–278.

Choi, Man-Duen. “Completely Positive Linear Maps on Complex Matrices.” Linear Algebra and Its Applications 10, no. 3 (1975): 285–290.

Kraus, Karl. “General State Changes in Quantum Theory.” Annals of Physics 64, no. 2 (1971): 311–335.

Nielsen, Michael A., and Isaac L. Chuang. Quantum Computation and Quantum Information. Cambridge University Press, 10th anniversary edition, 2010.

Watrous, John. The Theory of Quantum Information. Cambridge University Press, 2018.

Wilde, Mark M. Quantum Information Theory. Cambridge University Press, 2nd edition, 2017.

Paulsen, Vern. Completely Bounded Maps and Operator Algebras. Cambridge University Press, 2003.

Document information

Choi-Jamiołkowski Isomorphism

Project	[QIT 001] Mathematical Representation Theorems
Document	Primary document
Author	mujirin
Verifier	Not verified
Downloaded	July 04, 2026 18:13 KST
Status	Working
Document link	https://www.theorytrace.com/projects/mathematical-representation-theorems/documents/untitled-document-e38927/