Stinespring Dilation Theorem

Formal statement

Let mathcal H(A) and mathcal H(B) be finite-dimensional complex Hilbert spaces, and let

$$\Phi : L(\mathcal{H}_A) \to L(\mathcal{H}_B)$$

be a completely positive linear map. In finite-dimensional quantum information, the most important case is when Φ is also trace preserving, in which case Φ is called a quantum channel.

The Stinespring dilation theorem says that if Φ is completely positive, then there exists an auxiliary Hilbert space mathcal H(E), called an environment, and a linear operator

$$V : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$$

such that

$$\Phi(\rho) = \mathrm{Tr}_E \left(V \rho V^\dagger \right)$$

for every operator ρ∈ L(mathcal H(A)). If Φ is trace preserving, then V may be chosen to be an isometry:

$$V^\dagger V = I_A.$$

Equivalently, every finite-dimensional quantum channel can be implemented by preparing an environment in a fixed pure state, applying a unitary to the system plus environment, and then discarding the environment:

$$\Phi(\rho) = \mathrm{Tr}_E \left[U(\rho \otimes |0\rangle\langle 0|_E) U^\dagger \right].$$

A small correction to the common informal statement is important. A general completely positive map need not preserve trace, so it does not always describe deterministic physical evolution by unitary interaction followed only by discarding an environment. The clean unitary-and-discard form is for completely positive trace-preserving maps. General trace-nonincreasing completely positive maps describe probabilistic branches of an operation, such as one measurement outcome.

The theorem originates from Stinespring's 1955 work on positive functions on C^-algebras. In quantum information, it is one of the standard equivalent descriptions of quantum channels, alongside the Kraus/operator-sum representation and the Choi representation.

Why this theorem matters

The theorem explains why nonunitary quantum evolution is still compatible with the unitary postulates of closed-system quantum mechanics. A closed system evolves unitarily. An open system may appear to evolve nonunitarily because it interacts with an environment and we later ignore, lose, or discard that environment.

Thus, the Stinespring picture gives the following operational mental image. A channel is not a mysterious transformation acting only on A. It is the visible shadow of an ordinary unitary transformation on a larger closed system. The system A couples to an environment E, information may flow from A into E, and then we describe only the remaining output system B. The partial trace represents forgetting the environment.

This is why the theorem is structurally parallel to purification and Naimark dilation. Purification says that every mixed state is the marginal of a pure state. Naimark dilation says that every POVM is the compression of a projective measurement on a larger Hilbert space. Stinespring dilation says that every quantum channel is the reduction of a unitary or isometric evolution on a larger Hilbert space.

Proof from Kraus operators

In finite dimensions, the simplest proof uses the Kraus representation. If Φ is completely positive, then it admits an operator-sum representation

$$\Phi(\rho) = \sum_{k=1}^{m} A_k \rho A_k^\dagger,$$

where each

$$A_k : \mathcal{H}_A \to \mathcal{H}_B$$

is a linear operator. If Φ is trace preserving, then the Kraus operators satisfy

$$\sum_{k=1}^{m} A_k^\dagger A_k = I_A.$$

If Φ is trace nonincreasing, then

$$\sum_{k=1}^{m} A_k^\dagger A_k \le I_A.$$

Now introduce an environment Hilbert space mathcal H(E) with orthonormal basis

$$\{|k\rangle_E : k = 1, \dots, m\}.$$

Define

$$V : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$$

by

$$V|\psi\rangle = \sum_{k=1}^{m} A_k|\psi\rangle \otimes |k\rangle_E.$$

We first compute V^† V. For arbitrary vectors |ψ⟩,|φ⟩∈mathcal H(A),

$$\begin{aligned}
\langle V\varphi|V\psi\rangle &= \sum_{j,k} \langle\varphi|A_j^\dagger A_k|\psi\rangle\langle j|k\rangle_E \\
&= \sum_{k} \langle\varphi|A_k^\dagger A_k|\psi\rangle \\
&= \left\langle \varphi \middle| \left(\sum_k A_k^\dagger A_k \right) \middle| \psi \right\rangle.
\end{aligned}$$

Therefore

$$V^\dagger V = \sum_k A_k^\dagger A_k.$$

If Φ is trace preserving, this gives

$$V^\dagger V = I_A,$$

so V is an isometry. If Φ is trace nonincreasing, then V^† V≤ I(A), so V is a contraction on the input space.

Now compute the partial trace over the environment:

$$V\rho V^\dagger = \left(\sum_k A_k \rho^{1/2} \otimes |k\rangle_E\right)\left(\sum_j \rho^{1/2} A_j^\dagger \otimes \langle j|_E\right)$$

or, more directly,

$$V\rho V^\dagger = \sum_{j,k} A_k \rho A_j^\dagger \otimes |k\rangle\langle j|_E.$$

Taking the partial trace over E, we use

$$\mathrm{Tr}_E(|k\rangle\langle j|) = \delta_{kj}.$$

Hence

$$\begin{aligned}\mathrm{Tr}_E(V\rho V^\dagger) &= \sum_{j,k} A_k \rho A_j^\dagger \,\mathrm{Tr}(|k\rangle\langle j|)\\ &= \sum_k A_k \rho A_k^\dagger\\ &= \Phi(\rho).\end{aligned}$$

Thus every completely positive map has a Stinespring representation. When the map is trace preserving, the Stinespring operator V is an isometry.

From isometry to unitary interaction

The isometry form already proves the essential theorem. To obtain the more physical unitary form, assume Φ is trace preserving, so $V^\dagger V=I_{(A)}$. Choose an environment initialized in a fixed state $|0\rangle_{(E)}$. The map

$$|\psi\rangle_A|0\rangle_E \mapsto V|\psi\rangle_A$$

preserves inner products, because V is an isometry. Therefore it maps an orthonormal set to an orthonormal set. In finite dimensions, any such isometric action on a subspace can be extended to a unitary operator on a sufficiently large Hilbert space. Thus there exists a unitary U such that

$$U(|\psi\rangle_A \otimes |0\rangle_E) = V|\psi\rangle_A$$

for all $|\psi\rangle_{(A)}$. Consequently,

$$\Phi(\rho) = \mathrm{Tr}_E\left[U(\rho \otimes |0\rangle\langle 0|_E)U^\dagger\right].$$

This is the form most often meant in physics when people say that every quantum channel arises from unitary evolution on a larger system.

Operational interpretation

The theorem tells us that the most general deterministic evolution of an open quantum system has three conceptual steps. First, attach an environment in a known initial state. Second, let the system and environment evolve together unitarily. Third, discard the environment.

The environment is not necessarily a literal thermal bath. It may represent lost photons, unused modes, a measurement apparatus, a noise source, an inaccessible reference system, or simply a mathematical workspace. The theorem says that whatever completely positive trace-preserving transformation we write down as a quantum channel, there is always some larger closed-system story that realizes it.

This gives a precise meaning to information leakage. If the channel is unitary on the system alone, no environment is needed. If the channel is noisy, then in the dilation some information about the input has flowed into the environment. The output system B may retain part of the input information, while the environment retains the complementary part. This complementary environment output is the basis of complementary channels, coherent information, decoupling, and quantum error correction.

Example 1: a unitary channel needs a trivial environment

Let

$$\Phi(\rho) = U\rho U^\dagger$$

for a unitary U:mathcal H(A)→mathcal H(B). This channel has one Kraus operator:

$$A_1 = U.$$

The environment can be one-dimensional. The Stinespring isometry is simply

$$V|\psi\rangle = U|\psi\rangle \otimes |0\rangle_E.$$

Tracing out the one-dimensional environment changes nothing:

$$\mathrm{Tr}_E(V\rho V^\dagger) = U\rho U^\dagger.$$

This example shows that unitary dynamics is the special case of Stinespring dilation in which the environment carries no nontrivial information.

Example 2: complete dephasing as information copied into the environment

Consider the qubit dephasing channel

$$\Delta(\rho) = |0\rangle\langle 0|\rho|0\rangle\langle 0| + |1\rangle\langle 1|\rho|1\rangle\langle 1|.$$

Its Kraus operators are

$$A_0 = |0\rangle\langle 0|, \qquad A_1 = |1\rangle\langle 1|.$$

The corresponding Stinespring isometry is

$$V|\psi\rangle = A_0|\psi\rangle|0\rangle_E + A_1|\psi\rangle|1\rangle_E.$$

If

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle,$$

then

$$V|\psi\rangle = \alpha|0\rangle|0\rangle_E + \beta|1\rangle|1\rangle_E.$$

Now trace out the environment:

$$\mathrm{Tr}_E(V|\psi\rangle\langle\psi|V^\dagger) = |\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|.$$

The off-diagonal coherence

$$\alpha\overline{\beta}|0\rangle\langle 1| + \overline{\alpha}\beta|1\rangle\langle 0|$$

has disappeared from the system because the environment has become correlated with the computational-basis label. Operationally, dephasing is not magic destruction of coherence. It is coherence becoming inaccessible because which-basis information has leaked into the environment.

Example 3: erasure channel and a visible flag

The erasure channel sends a state through unchanged with probability 1-p, and replaces it with an orthogonal erasure flag |e⟩ with probability p. For a qubit input,

$$\Phi(\rho) = (1-p)\rho + p|e\rangle\langle e|,$$

where the output Hilbert space is enlarged so that |e⟩ is orthogonal to |0⟩ and |1⟩.

One Kraus representation is

$$A_0 = \sqrt{1-p}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right),$$

$$A_1 = \sqrt{p}|e\rangle\langle 0|, \qquad A_2 = \sqrt{p}|e\rangle\langle 1|.$$

These satisfy

$$A_0^\dagger A_0 + A_1^\dagger A_1 + A_2^\dagger A_2 = I.$$

The Stinespring isometry is

$$V|\psi\rangle = A_0|\psi\rangle|0\rangle_E + A_1|\psi\rangle|1\rangle_E + A_2|\psi\rangle|2\rangle_E.$$

The environment records which physical branch occurred. The receiver sees either the original quantum system or the erasure flag. This example is important because it shows that the output space mathcal H(B) need not be the same as the input space mathcal H(A).

Example 4: amplitude damping as energy leaking to an environment

The qubit amplitude damping channel models spontaneous decay from |1⟩ to |0⟩ with probability γ. A common Kraus representation is

$$A_0 = |0\rangle\langle 0| + \sqrt{1-\gamma}|1\rangle\langle 1|,$$

$$A_1 = \sqrt{\gamma}|0\rangle\langle 1|.$$

The Stinespring isometry is

$$V|\psi\rangle = A_0|\psi\rangle|0\rangle_E + A_1|\psi\rangle|1\rangle_E.$$

On basis states,

$$V|0\rangle = |0\rangle|0\rangle_E,$$

and

$$V|1\rangle = \sqrt{1-\gamma}|1\rangle|0\rangle_E + \sqrt{\gamma}|0\rangle|1\rangle_E.$$

This is the operational picture of decay. If the qubit is already in the ground state $|0\rangle$, nothing happens. If it is excited, part of its amplitude remains excited with no emitted environmental excitation, and part decays to $|0\rangle$ while the environment records an emitted quantum. Tracing out the environment gives the familiar irreversible damping channel.

Example 5: a measurement outcome as a non-trace-preserving CP map

Let

$$M_0 = |0\rangle\langle 0|$$

be the operation corresponding to obtaining outcome 0 in a computational-basis measurement. The map

$$\Phi_0(\rho) = M_0 \rho M_0^\dagger$$

is completely positive, but it is not trace preserving. Its trace is

$$\operatorname{Tr}\Phi_0(\rho) = \operatorname{Tr}(M_0^\dagger M_0 \rho) = \langle 0|\rho|0\rangle,$$

which is the probability of outcome 0. This map describes one probabilistic branch, not a deterministic channel.

This example explains why the informal statement “every completely positive map is unitary plus discard” needs qualification. If the trace decreases, the missing trace is an outcome probability. To realize such a map physically, one usually performs a larger trace-preserving operation and then conditions on a measurement outcome. The deterministic object is the full instrument, while each branch is a trace-nonincreasing completely positive map.

The complementary channel

Once a Stinespring isometry

$$V : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$$

has been chosen, the ordinary channel to the receiver is

$$\Phi(\rho) = \mathrm{Tr}_E(V\rho V^\dagger).$$

The complementary channel is obtained by tracing out the receiver instead:

$$\Phi^c(\rho) = \mathrm{Tr}_B(V\rho V^\dagger).$$

The complementary channel describes what the environment learns. This is not a side detail. Many deep ideas in quantum information are statements about the balance between what reaches the receiver and what leaks to the environment. Quantum error correction, private communication, coherent information, and decoupling all use this system-environment split.

For example, a noise process is correctable on a code when the environment cannot distinguish the encoded states too well. In that case, information has not leaked irreversibly to the environment, and a recovery operation can restore the system.

How to use the theorem in calculations

When given a channel in Kraus form,

$$\Phi(\rho) = \sum_k A_k \rho A_k^\dagger,$$

construct the Stinespring isometry by attaching an environment label to each Kraus operator:

$$V = \sum_k A_k \otimes |k\rangle_E.$$

This compact expression means

$$V|\psi\rangle = \sum_k A_k|\psi\rangle \otimes |k\rangle_E.$$

Then the channel is recovered by tracing out E:

$$\Phi(\rho) = \mathrm{Tr}_E(V\rho V^\dagger).$$

If the channel is trace preserving, then $\sum_k A_k^\dagger A_k = I$, so V is an isometry. If one wants a circuit implementation, extend V to a unitary on the system plus environment. The number of Kraus operators gives an upper bound on the environment dimension needed for this construction. A minimal Stinespring environment has dimension equal to the minimal number of Kraus operators, which is the rank of the Choi matrix of the channel.

Relation to Kraus and Choi representations

The Stinespring, Kraus, and Choi pictures are three different ways to describe the same finite-dimensional object.

The Kraus representation says that a channel is a sum of elementary branches:

$$\Phi(\rho) = \sum_k A_k \rho A_k^\dagger.$$

The Stinespring representation says that these branches can be coherently stored in an environment:

$$V|\psi\rangle = \sum_k A_k|\psi\rangle|k\rangle_E.$$

The Choi representation stores the entire channel in a bipartite operator

$$J(\Phi) = (\mathrm{id} \otimes \Phi)(|\Omega\rangle\langle\Omega|),$$

where $|\Omega\rangle$ is an unnormalized maximally entangled vector. Complete positivity of Φ is equivalent to positivity of $J(\Phi)$. The rank of this Choi operator equals the minimal number of Kraus operators, and therefore also the minimal Stinespring environment dimension in finite dimensions.

Operationally, Kraus operators describe the branches, Stinespring dilation describes the coherent system-environment mechanism, and the Choi matrix describes the channel as a state-like object.

Common mistakes

A common mistake is to forget trace preservation. Complete positivity alone gives a Stinespring-type operator representation, but the clean deterministic unitary interaction followed by discarding an environment corresponds to completely positive trace-preserving maps. A trace-decreasing CP map represents a conditional branch, not a full deterministic evolution.

A second mistake is to think the environment is unique. It is not. Different Kraus representations give different Stinespring isometries. Minimal Stinespring dilations are unique only up to an isometry or unitary on the environment, analogous to uniqueness of purification.

A third mistake is to treat the Kraus index as a classical random variable too early. In the Stinespring isometry, the environment label $|k\rangle_{(E)}$ is stored coherently. Only after tracing out or measuring the environment does it behave like an inaccessible classical branch. This distinction matters in coherent information and quantum error correction.

A fourth mistake is to think that all loss of coherence means fundamental nonunitarity. In the Stinespring picture, the total system plus environment remains coherent and unitary. The apparent nonunitarity appears because we look only at a subsystem.

Final mental image

The Stinespring dilation theorem is the structure theorem for quantum channels. It says that every open-system evolution can be understood as closed-system unitary evolution on a larger Hilbert space, followed by forgetting part of that larger system.

In one line,

$$\Phi(\rho) = \mathrm{Tr}_E(V\rho V^\dagger),$$

and, for a channel,

$$\Phi(\rho) = \mathrm{Tr}_E\left[U(\rho \otimes |0\rangle\langle 0|_E)U^\dagger\right].$$

The environment is where the missing information goes. If no information leaks to the environment, the channel is essentially unitary. If information does leak, the system appears noisy. Thus Stinespring dilation gives the operational meaning of noise: noise is entanglement and correlation with degrees of freedom we do not keep.

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