Unitary Freedom of Kraus Representations

Formal statement

Let mathcal H$_{(A)}$ and mathcal H$_{(B)}$ be finite-dimensional complex Hilbert spaces, and let

$$\mathcal{N} : L(\mathcal{H}_A) \to L(\mathcal{H}_B)$$

be a completely positive map. In particular, if mathcal N is trace preserving, then it is a quantum channel. Suppose one Kraus representation of mathcal N is

$$\mathcal{N}(\rho) = \sum_{i=1}^{r} E_i \rho E_i^\dagger,$$

and another Kraus representation is

$$\mathcal{N}(\rho) = \sum_{j=1}^{n} F_j \rho F_j^\dagger.$$

The cleanest version of the theorem is the following.

If $E_i\backslash_{i=1}^{r}$ is a minimal Kraus representation, meaning that the operators $E_1,\dots,E_r$ are linearly independent, then every other Kraus representation $F_j\backslash_{j=1}^{n}$ of the same map is obtained by an isometry on the Kraus index space. More explicitly, there exists an $n\times r$ matrix

$$U = (u_{ji})$$

satisfying

$$U^\dagger U = I_r$$

such that

$$F_j = \sum_{i=1}^{r} u_{ji} E_i$$

for every j=1,...,n. The condition U^† U=I_r means that U:mathbb C^r→mathbb C^n is an isometry.

If both Kraus representations are minimal, then n=r, and the isometry becomes a unitary matrix:

$$U^\dagger U = UU^\dagger = I_r.$$

Thus two minimal Kraus representations of the same channel differ only by a unitary mixing of their Kraus operators.

If neither representation is minimal, then the safest statement is this: after padding the shorter Kraus list with zero operators so that both lists have the same length, the two lists are related by a unitary matrix on the enlarged Kraus index space.

This theorem is sometimes called the unitary freedom of Kraus operators. The word “unitary” is exact for minimal representations or equal-length padded representations. The more general unpadded statement uses an isometry.

Why this theorem matters

A quantum channel is a physical transformation. A Kraus representation is not itself unique physics. It is a choice of coordinates for describing how the channel can be decomposed into operator branches.

This distinction is important. When we write

$$\mathcal{N}(\rho) = \sum_k E_k \rho E_k^\dagger,$$

it is tempting to imagine that the channel literally chooses one branch k with some probability. Sometimes that interpretation is possible, especially when a particular environment is measured in a particular basis. But the Kraus representation itself does not uniquely define such classical alternatives. Another equally valid Kraus representation can mix the operators E_k by a unitary or isometry and produce the same channel.

The operational mental image is this. In the Stinespring picture, Kraus operators arise by choosing an orthonormal basis for the environment. Changing the environment basis changes the Kraus operators, but it does not change the reduced dynamics of the system after the environment is discarded. Therefore the Kraus index is like an environment coordinate. It is not an invariant label unless the physical measurement apparatus selects that basis.

The theorem is the mathematical expression of this freedom.

Proof using the Choi matrix

We give a finite-dimensional proof. The key idea is that a Kraus representation is a way of decomposing the Choi matrix into rank-one positive terms. Two decompositions of the same positive operator differ by an isometry on the decomposition index.

For any operator E_i:mathcal $H_{(A)}$→mathcal $H_{(B)}$, define its vectorization by

$$|E_i\rangle\rangle = (I_{A'} \otimes E_i)|\Omega\rangle_{A'A},$$

where

$$|\Omega\rangle_{A'A} = \sum_a |a\rangle_{A'}|a\rangle_A$$

is the unnormalized maximally entangled vector associated with a chosen basis of mathcal $H_{(A)}$.

If

$$\mathcal{N}(\rho) = \sum_i E_i \rho E_i^\dagger,$$

then its Choi matrix is

$$J(\mathcal{N}) = \sum_i |E_i\rangle\rangle\langle\langle E_i|.$$

Likewise, if

$$\mathcal{N}(\rho) = \sum_j F_j \rho F_j^\dagger,$$

then

$$J(\mathcal{N}) = \sum_j |F_j\rangle\rangle\langle\langle F_j|.$$

Now assume $E_i\backslash_{i=1}{}^r$ is minimal. This means that the vectors

$$|E_1\rangle\rangle, \dots, |E_r\rangle\rangle$$

are linearly independent. Hence they form a basis for the support of J(mathcal N). Since the F_j's give the same Choi matrix, each $|F_j\rangle\rangle$ lies in the same support. Therefore there exist coefficients u_{ji} such that

$$|F_j\rangle\rangle = \sum_{i=1}^r u_{ji} |E_i\rangle\rangle.$$

Equivalently,

$$F_j = \sum_{i=1}^r u_{ji} E_i.$$

It remains to prove that the coefficient matrix $U=(u_{ji})$ is an isometry.

Substitute the relation $|F_j\rangle\rangle=\sum_i u_{ji}|E_i\rangle\rangle$ into the second Choi decomposition:

$$\begin{aligned} J(\mathcal{N}) &= \sum_j |F_j\rangle\rangle\langle\langle F_j| \\ &= \sum_j \sum_{i,\ell} u_{ji}\overline{u_{j\ell}} |E_i\rangle\rangle\langle\langle E_\ell| \\ &= \sum_{i,\ell} \left(\sum_j u_{ji}\overline{u_{j\ell}} \right) |E_i\rangle\rangle\langle\langle E_\ell|. \end{aligned}$$

But the original Choi decomposition is

$$J(\mathcal{N}) = \sum_i |E_i\rangle\rangle\langle\langle E_i|.$$

Because the $|E_i\rangle\rangle$'s are linearly independent, the rank-one operators

$$|E_i\rangle\rangle\langle\langle E_\ell|$$

are linearly independent. Therefore the coefficients must agree:

$$\sum_j u_{ji}\overline{u_{j\ell}} = \delta_{i\ell}.$$

This is exactly

$$U^\dagger U = I_r.$$

So U is an isometry, and

$$F_j = \sum_i u_{ji} E_i.$$

This proves the theorem when the first representation is minimal. If both representations are minimal, then both have the same number of Kraus operators, equal to the rank of the Choi matrix. Thus U is a square isometry, hence a unitary.

If a representation is not minimal, one can remove linear dependencies to reach a minimal representation, or equivalently pad Kraus lists with zeros and use a unitary relation on the padded index space. This is the general unitary freedom statement.

Converse direction

The converse is also important. Suppose

$$F_j = \sum_i u_{ji} E_i$$

and

$$U^\dagger U = I.$$

Then

$$\begin{aligned}\sum_j F_j\rho F_j^\dagger &= \sum_j \left(\sum_i u_{ji}E_i\right)\rho\left(\sum_\ell u_{j\ell}E_\ell\right)^\dagger \\ &= \sum_{j,i,\ell} u_{ji}\overline{u_{j\ell}}E_i\rho E_\ell^\dagger \\ &= \sum_{i,\ell}\left(\sum_j u_{ji}\overline{u_{j\ell}}\right)E_i\rho E_\ell^\dagger \\ &= \sum_{i,\ell}\delta_{i\ell}E_i\rho E_\ell^\dagger \\ &= \sum_i E_i\rho E_i^\dagger.\end{aligned}$$

Thus isometric mixing of Kraus operators does not change the channel. This proves not only necessity but also sufficiency.

Proof using Stinespring dilation

There is another proof that gives a better physical picture.

Given Kraus operators E_i, define the Stinespring isometry

$$V_E|\psi\rangle = \sum_i E_i|\psi\rangle \otimes |i\rangle_E.$$

Given Kraus operators F_j, define another Stinespring isometry

$$V_F|\psi\rangle = \sum_j F_j|\psi\rangle \otimes |j\rangle_F.$$

Both isometries realize the same channel after tracing out their environments:

$$\mathrm{Tr}_E(V_E\rho V_E^\dagger) = \mathcal{N}(\rho) = \mathrm{Tr}_F(V_F\rho V_F^\dagger).$$

The uniqueness theorem for minimal Stinespring dilations says that two minimal dilations of the same channel are related by a unitary on the environment, and a nonminimal dilation is related to a minimal one by an isometry into the larger environment. Since Kraus operators are obtained by expanding the Stinespring isometry in an environment basis, this environment isometry becomes exactly the isometry mixing the Kraus operators.

This proof explains why the theorem is physically natural. The Kraus index labels an environment basis. Changing the environment basis changes the Kraus operators but not the channel.

Example 1: a unitary channel and phase freedom

Consider the unitary channel

$$\mathcal{U}(\rho) = U\rho U^\dagger.$$

A minimal Kraus representation has one Kraus operator:

$$E_1 = U.$$

Another one-operator representation is

$$F_1 = e^{i\theta}U.$$

The channel is unchanged because

$$F_1\rho F_1^\dagger = e^{i\theta}U\rho e^{-i\theta}U^\dagger = U\rho U^\dagger.$$

Here the unitary freedom on the one-dimensional Kraus index space is just a phase:

$$U_{extindex} = (e^{i\theta}).$$

This example shows the simplest form of the theorem. A global phase on a Kraus operator has no physical effect on the channel.

Example 2: a nonminimal representation of a unitary channel

The same unitary channel can be written nonminimally as

$$\mathcal{U}(\rho) = \frac{1}{2}U\rho U^\dagger + \frac{1}{2}U\rho U^\dagger.$$

This corresponds to Kraus operators

$$F_1 = \frac{1}{\sqrt{2}}U, \qquad F_2 = \frac{1}{\sqrt{2}}U.$$

These are not linearly independent, so the representation is not minimal. The minimal representation $E_1=U$ is related to this two-operator representation by the isometry

$$\mathbb{C} \to \mathbb{C}^2, \qquad 1 \mapsto \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}.$$

Indeed,

$$F_j = u_{j1}E_1.$$

The channel has not become physically random merely because we wrote it using two Kraus operators. The two terms are a redundant description of the same unitary evolution.

This is the first important warning: the number of Kraus operators in a nonminimal representation is not an invariant property of the channel. The invariant number is the minimal Kraus number, equal to the rank of the Choi matrix.

Example 3: bit-flip channel and rotated Kraus operators

The bit-flip channel is

$$\mathcal{N}(\rho) = (1-p)\rho + pX\rho X.$$

A natural Kraus representation is

$$E_0 = \sqrt{1-p}\, I, \qquad E_1 = \sqrt{p}\, X.$$

For 0<p<1, this representation is minimal because I and X are linearly independent. Now choose the unitary matrix

$$U_{extindex} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Define

$$F_0 = \frac{E_0 + E_1}{\sqrt{2}}, \qquad F_1 = \frac{E_0 - E_1}{\sqrt{2}}.$$

Then

$$\sum_{j=0}^{1} F_j \rho F_j^\dagger = \sum_{i=0}^{1} E_i \rho E_i^\dagger.$$

The new Kraus operators are not simply “no flip” and “flip.” They are coherent linear combinations of those two branches. Yet they describe exactly the same channel.

This example shows why one must be cautious about interpreting a Kraus representation as a classical random story. The representation \√(1-p)I,sqrt p X\ admits a natural random-error interpretation. The rotated representation F_0,F_1\ describes the same channel, but its individual branches no longer correspond to the same classical alternatives.

Example 4: amplitude damping and the environment-basis picture

The amplitude damping channel has Kraus operators

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \qquad E_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}.$$

The usual interpretation is that E_0 is the no-jump branch and E_1 is the decay branch. The corresponding Stinespring isometry is

$$V|\psi\rangle = E_0|\psi\rangle|0\rangle_E + E_1|\psi\rangle|1\rangle_E.$$

Now rotate the environment basis by a Hadamard transformation:

$$|+\rangle_E = \frac{|0\rangle_E + |1\rangle_E}{\sqrt{2}}, \qquad |-\rangle_E = \frac{|0\rangle_E - |1\rangle_E}{\sqrt{2}}.$$

In this new environment basis, the same isometry can be written as

$$V|\psi\rangle = F_+|\psi\rangle|+\rangle_E + F_-|\psi\rangle|-\rangle_E,$$

where

$$F_+ = \frac{E_0 + E_1}{\sqrt{2}}, \qquad F_- = \frac{E_0 - E_1}{\sqrt{2}}.$$

The channel is unchanged. What changed is only the basis in which the environment label was read.

This is the operational heart of the theorem. Kraus freedom is environment-basis freedom.

Example 5: padding with zero Kraus operators

Suppose a channel has a minimal representation

$$\mathcal{N}(\rho) = E_1\rho E_1^\dagger + E_2\rho E_2^\dagger.$$

We may pad it with a zero Kraus operator:

$$\mathcal{N}(\rho) = E_1\rho E_1^\dagger + E_2\rho E_2^\dagger + 0\rho 0^\dagger.$$

Now any 3×3 unitary matrix can rotate the padded list

$$(E_1, E_2, 0)$$

into another three-operator list

$$(F_1, F_2, F_3).$$

The new list may have three nonzero operators, even though the channel has minimal Kraus number two. Thus a channel may admit many longer Kraus representations. Extra Kraus operators can be artifacts of a nonminimal environment.

This example explains the precise role of padding. When two Kraus lists have different lengths, we may add zero operators until their lengths match. Then the relation becomes unitary on the padded index space.

Minimal Kraus number and Choi rank

The theorem also explains why the minimal number of Kraus operators is well defined. Although Kraus representations are not unique, all minimal representations have the same length. That length is

$$\operatorname{rank} J(\mathcal{N}),$$

where J(mathcal N) is the Choi matrix of the channel.

Indeed, every Kraus representation gives a decomposition

$$J(\mathcal{N}) = \sum_k |E_k\rangle\rangle\langle\langle E_k|.$$

The smallest number of rank-one terms needed to decompose a positive semidefinite operator is its rank. Therefore the minimal number of Kraus operators is the rank of the Choi matrix.

This invariant is often called the Choi rank or Kraus rank. It is also the minimal environment dimension needed in a Stinespring dilation.

Common mistakes

A common mistake is to say that Kraus operators of a channel are unique. They are not. Only the channel is unique. The Kraus representation depends on a choice of environment basis or a choice of rank-one decomposition of the Choi matrix.

A second mistake is to interpret every Kraus index as a physical classical event. That interpretation is representation-dependent. A special Kraus representation may correspond to a physically monitored environment, but a unitary rotation of the Kraus operators gives the same channel with different branches.

A third mistake is to say “unitary” when “isometry” is needed. If one representation is minimal and the other is longer, the coefficient matrix is rectangular, so it cannot be unitary. It is an isometry from the minimal Kraus index space into the larger one. If both lists are padded to the same length, the relation can be made unitary.

A fourth mistake is to confuse nonminimal Kraus number with physical complexity. A unitary channel can be written with many redundant Kraus operators by splitting the same operator into several pieces. The minimal Kraus number, not an arbitrary displayed number of Kraus operators, captures the intrinsic environment dimension.

Final mental image

A Kraus representation is a choice of coordinates for the environment branch space. If two Kraus representations describe the same channel, they are two different coordinate systems for the same environmental information.

In the minimal case, the relation is a unitary rotation:

$$F_j = \sum_i u_{ji} E_i.$$

In the nonminimal case, the minimal branch space is embedded into a larger redundant branch space by an isometry. Padding with zero operators converts the statement into a unitary relation on a larger index space.

So the theorem can be remembered as follows:

$$\text{same channel} \quad \Longrightarrow \quad \text{same Kraus operators up to environment-basis isometry.}$$

This is why Kraus operators are powerful but must be interpreted carefully. The channel is physical. A Kraus representation is a representation. Its freedom is exactly the freedom to rotate, relabel, or enlarge the hidden environment basis without changing the system dynamics.

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